Name: ______________________________

Day: ______________________________

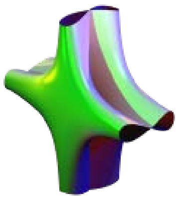
ImaginaryBCN

Monitoring notebook

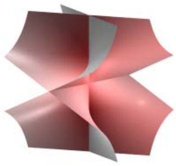
Activity: Equations and Singularities I

◆ Forms and equations from the exhibition

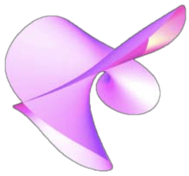
Which of these figures contains the point $(\mathbf{1}, \mathbf{1}, \mathbf{1})$?



$6x^2 = 2x^4 + y^2z^2$

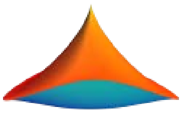


$x^2 - y^2z^2 = 0$

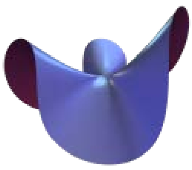


$x^2 = y^2z^2 + z^3$

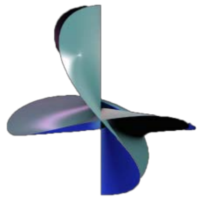
Which of these equations has the smallest degree?



$(xy - z^3 - 1)^2 = (1 - x^2 - y^2)^3$

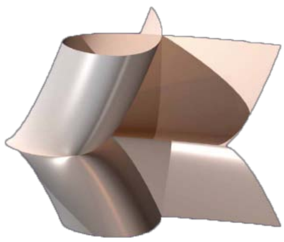


$x^3 + y^2z^3 + yz^4 = 0$



$x^2yz + xy^2 + y^3 + y^3x = x^2z^2$

Which is the coefficient of the monomial of highest degree of *Quaste*?



$$8z^9 - 24x^2z^6 - 24y^2z^6 + 36z^8 + 24x^4z^3 - 168x^2y^2z^3 + 24y^4z^3 - 72x^2z^5 - 72y^2z^5 + 54z^7 - 8x^6 - 24x^4y^2 - 24x^2y^4 - 8y^6 + 36x^4z^2 - 252x^2y^2z^2 + 36y^4z^2 - 54x^2z^4 - 108y^2z^4 + 27z^6 - 108x^2y^2z + 54y^4z - 54y^2z^3 + 27y^4 = 0$$

Material created by Maria Alberich, Jordi Buendía, Ferran Dachs, Anna Sabater and Emilio José Sánchez supported by:

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Universitat Politècnica de Catalunya BarcelonaTech
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fMe

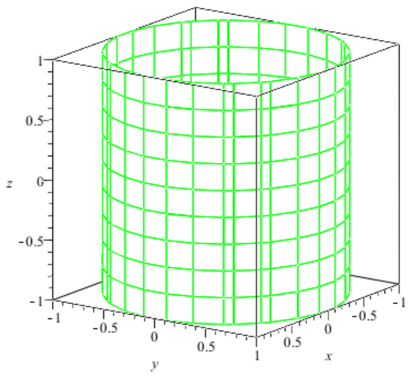
Form and formula

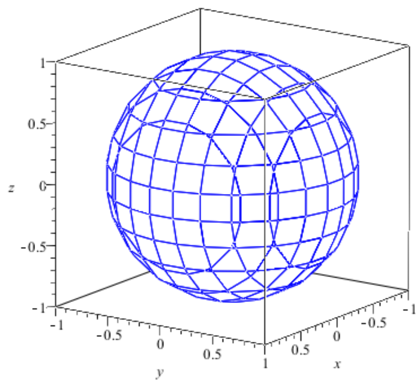
Equations to draw figures

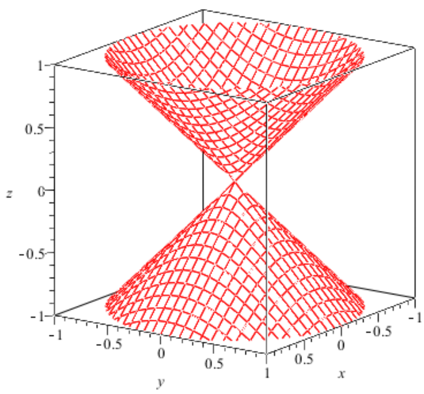
The solutions of an equation determine a set of points in the space, which make the figure. In the plane, $x^2 + y^2 = R^2$ is the equation of a circumference of radius R. Which figure in the space corresponds to each equation?

$$x^2 + y^2 = 1 \qquad x^2 + y^2 = z^2 \qquad x^2 + y^2 + z^2 = 1$$

Try to cut the figures with horizontal planes doing the following: for example, if you want to cut a figure with the horizontal planes $z = 0$, $z = \frac{1}{2}$ or $z = 1$ you have to replace the variable z in the equation by $0, 1/2$ or 1, respectively, and analyze the curve that you obtain.

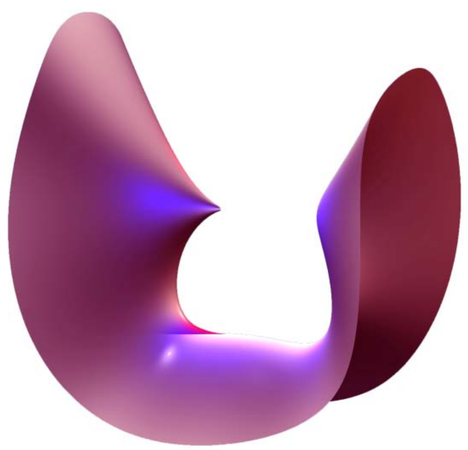






Finally, you can display them using SURFER! You must remember to enter the equation in the SURFER bar properly, this is, equated to zero. For example, if you want to display $x^2 + y^2 = 1$, you will have to write $x\^2 + y\^2 - 1$.

Singularities



The singular points, or singularities, can be identified visually, as the surface is not smooth at these points, for example, a sharp point or a fold. The peak on the left of Vis-à-Vis surface is a singularity, but the smooth mountain on the right has only regular points.

You may be able to recognize the singularities observing the surface carefully. But now imagine that you cannot touch or see the surface. Finding the singularities of the surface might be a challenge, isn't it? Here there is a mathematical trick: singularities are defined as those points on the surface where the partial derivatives of the equation are zero. The derivatives are useful to explain the behaviour of a function. This method helps you to find the singularities of the surface with only a piece paper and a pen, without seeing the surface, using only its equation.

Start with *Zitrus*' equation: $x^2 + z^2 = y^3(1-y)^3$. If you derive

$$f(x,y,z) = x^2 + z^2 - y^3(1-y)^3$$

and calculate the partial derivatives with respect to x, y and z, you get:

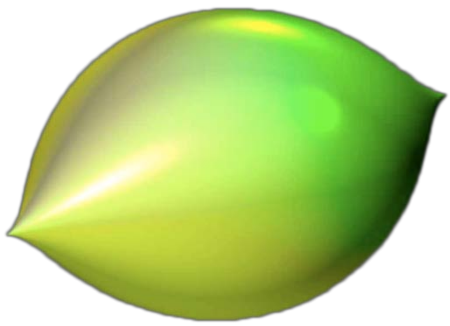
$$\frac{\partial f(x,y,z)}{\partial x} = 2x$$

$$\frac{\partial f(x,y,z)}{\partial y} = -3y^2(1-y)^3 + 3y^3(1-y)^2 = 3y^2(1-y)^2(2y-1)$$

$$\frac{\partial f(x,y,z)}{\partial z} = 2z$$

Now ask these partial derivatives to be zero. Then find the points (x, y, z) which are solutions of these three equations:

$$\begin{cases} 2x = 0 \\ 3y^2(1-y)^2(2y-1) = 0 \\ 2z = 0 \end{cases}$$



Which points did you found?

(, ,)

(, ,)

(, ,)

Check which of this points that you've found are singularities of *Zitrus*. Which of them belongs to the surface? Just check if the triplets are solution of the equation of *Zitrus*:

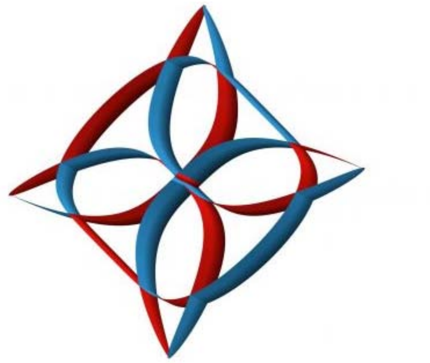
$$x^2 + z^2 = y^3(1-y)^3$$

Mark on the figure these points that are candidates to be a singularity. Looking the figure you had already guessed that the two spikes were singularities of Zitrus! What happens with the point $\left(0, \frac{1}{2}, 0\right)$? Is it also a singularity?

Material created by Maria Alberich, Jordi Buendía, Ferran Dachs, Anna Sabater and Emilio José Sánchez supported by:

◆ Create a surface with SURFER!

Invent an equation that has a low degree, for example 2, and another that has a high degree, for example 5. Check how are the surfaces that you have invented with SURFER. It is true that the equation of higher degree gives you a more complicated surface?



$0 = (x^2 + y^6 - 1) \cdot (2x^3 + 4y) \cdot (2y^3 + 2x) \cdot (2x^3 - 4y) \cdot (2y^3 - 2x)$

The quadratic cone and its singularity:

Enter at the SURFER gallery, get into "Simple singularities" gallery and choose the quadratic cone. Select it and look what happens varying the value of a:

- If you choose a value of a less than 0.5 you will see two separated pieces of the surface. This surface is called hyperboloid of two sheets.
- However, if you choose a value of a greater than 0.5, you will get a one-piece surface, called hyperboloid of one sheet.

Have you noticed? Small variations in the equation lead to three completely different surfaces, and the intermediate step from one to another is a singular surface!

Dare yourself to make changes:

As you know, if you write an equation of the form $ax + by + cz + d = 0$, you get a plane. What happens when you put a surface of degree 2? For example, $y - 2xz = 0$. Realize that the result is a curved surface. Now write $y - 200xz = 0$; what about the surface? And what if you have $100y - 2xz = 0$? As you have noticed, there is a monomial that bends the surface and another that makes the effect of flattening it. With only one high coefficient in the equation you can enhance the effects of other monomials.

Choose any surface of “Remarkable surfaces” gallery and transform it. For example, how can you change *Distel* to have only four spikes? Which coefficient must you increase, or which one must you remove?

Note that the figure above resembles to a gothic rose window. Dare yourself to design your own rose.

<u>Remember:</u>

Create your own surface and participate to the contest!

www.imaginary-exhibition.com/concurso

You can download (for free!) SURFER from the website:

www.imaginary-exhibition.com/surfer?lang=es