Sundials, Mathematics and Astronomy

Suzana Nápoles[1],
Margarida Oliveira[2]

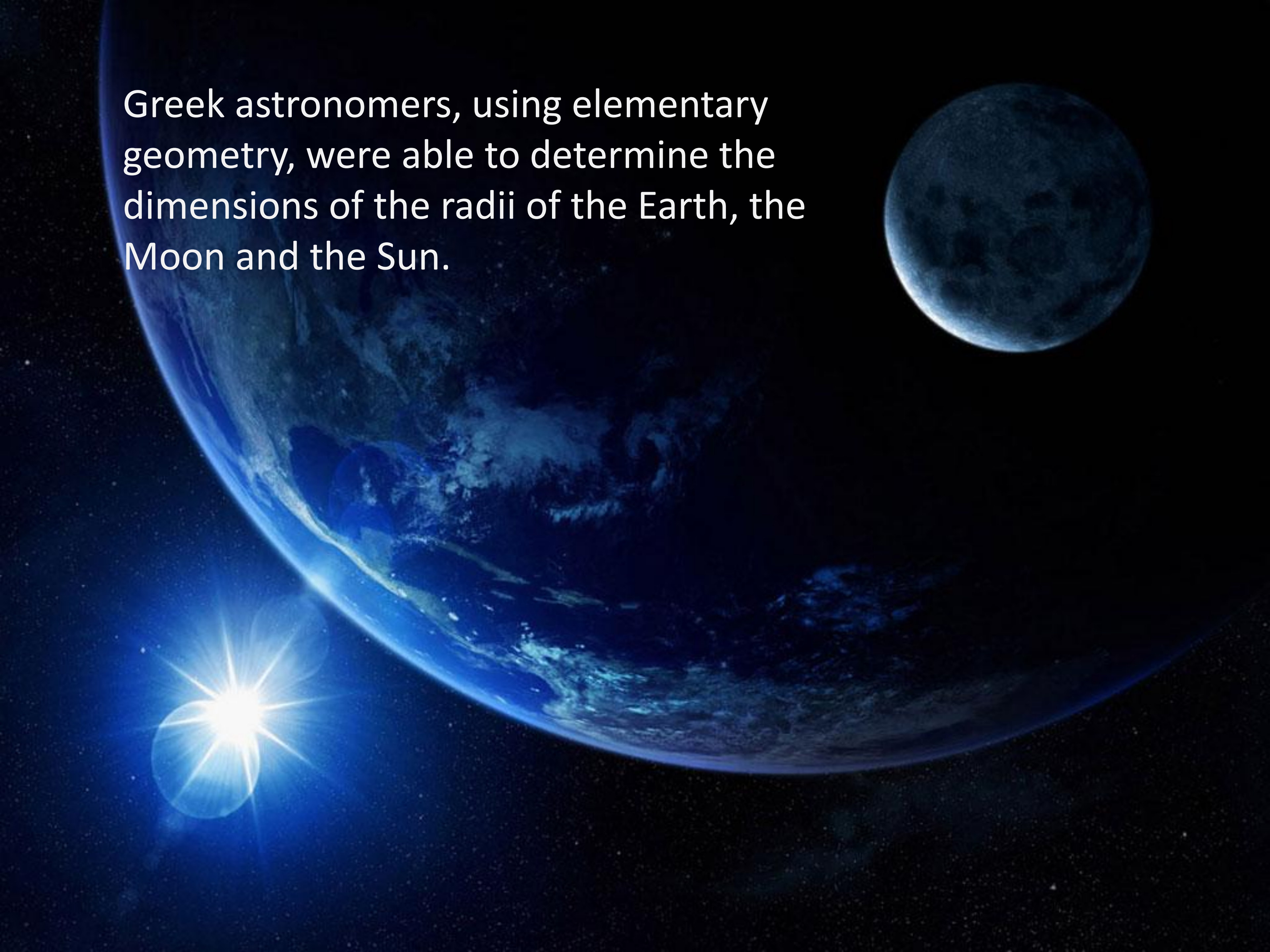
[1]Dep de Matemática, Faculdade de Ciências, Universidade de Lisboa (UL) , PORTUGAL

[2] Escola E.B. 2,3 Piscinas Lisboa; Centro de Matemática da Universidade dp Minho ,PORTUGAL

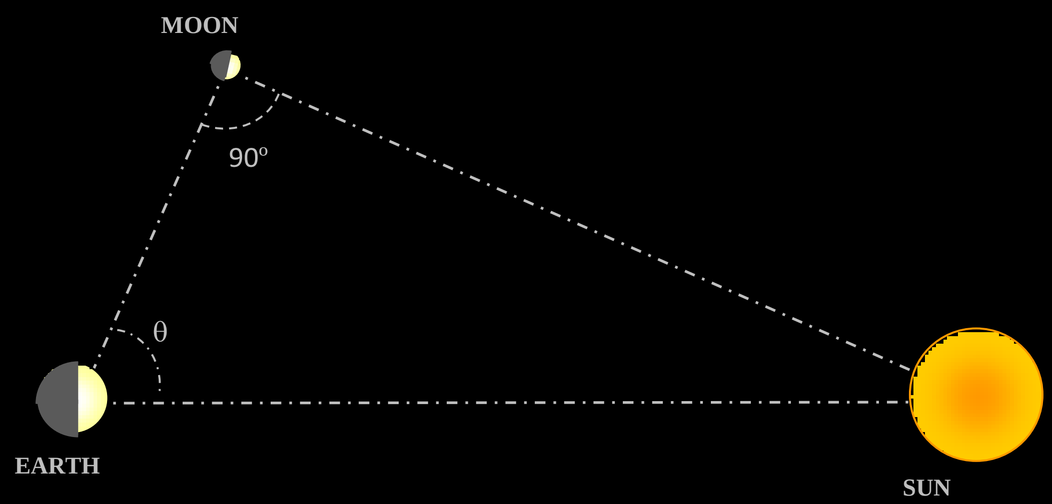
Relationship between mathematics and astronomy goes back in time.

An analysis thought superficial, of a few episodes in the history of Mathematics shows how this science is actually fundamental to the progress of the different branches of knowledge.

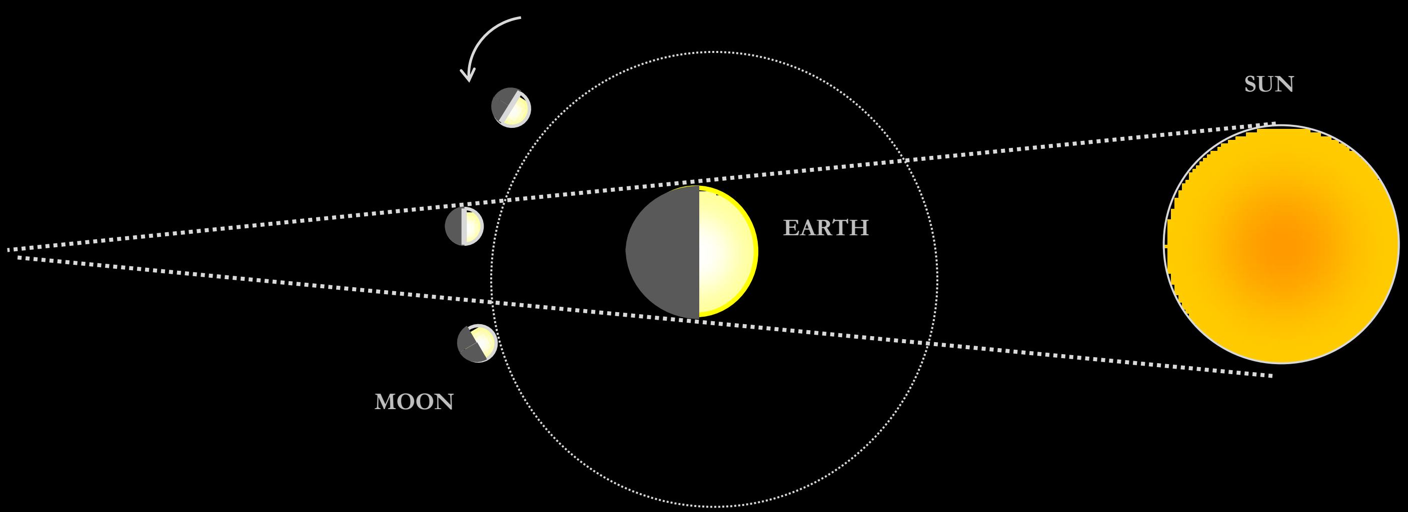
Greek astronomers, using elementary geometry, were able to determine the dimensions of the radii of the Earth, the Moon and the Sun.



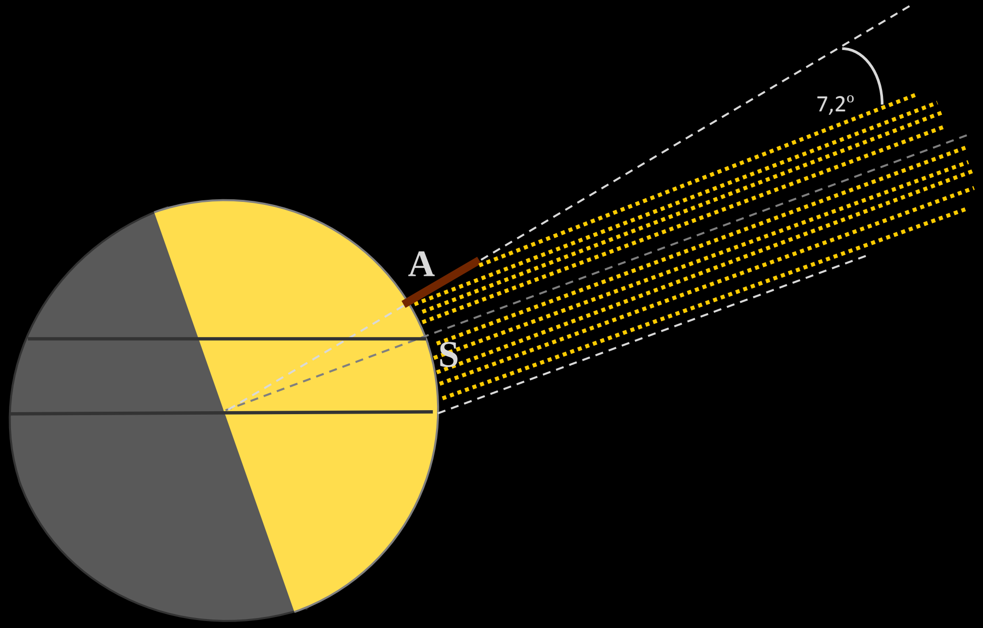
Three centuries before Christ, Aristarchus of Samos used the phases of the moon to measure the relative distances between the Earth and the Sun, the Earth and the Moon.



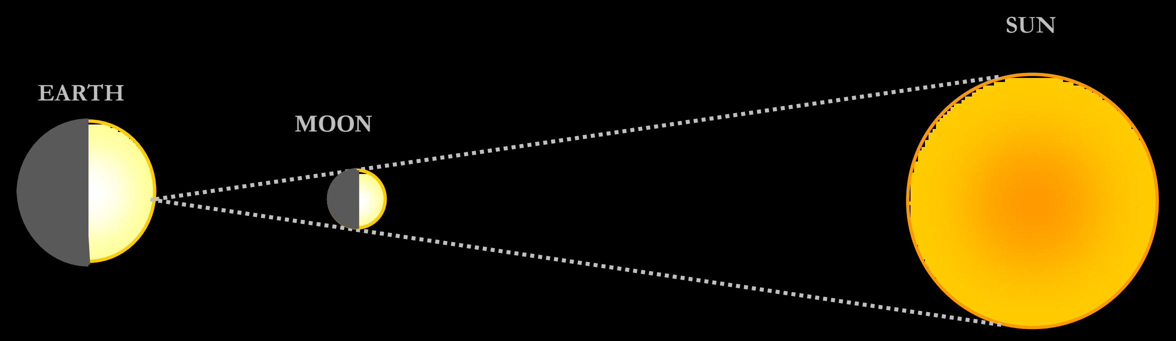
Using the moon eclipse Aristarchus compared the sizes of the Earth and the Moon.



Eratosthenes used the shadows of a column to measure the radius of the Earth.



Hipparchus used the Sun eclipse to compare the size of the Moon with the size of the Sun, allowing to compare de radii of those three celestial bodies.



The measurement of time also worried the sages of antiquity.

Questions like "What causes the variation in the direction of the object shadows during the day?"

and

"How can we take advantage of the variation in the direction of the shadows to measure time?"

were certainly present in the invention of the sundial.

With the advent of mechanical clocks, the question

“What is the relationship between the hours indicated by a sundial and by a mechanical clock at the same location?”

rose naturally.

The aim of this work is to explain how these questions have been answered over time.

In a way barely conscious, primitive man must have noticed that the Sun appears to move and that its luminosity varies, alternating periods of light with periods of darkness.

He must have also associated colder days with shorter periods of light and lower Sun, and warmer days with longer periods of luminosity and higher Sun.

The periods of luminosity were the origin of the measurement of time, since the temporary hours, taken as time measurement units, resulted in its division into 12 parts.

However, as the periods of light and darkness vary throughout the year, the temporary hours had a variable duration.

Thus, Hipparchus divided the set of the two periods in 24 equal hours, which later were divided by Ptolemy in 60 minutes and these in 60 seconds.

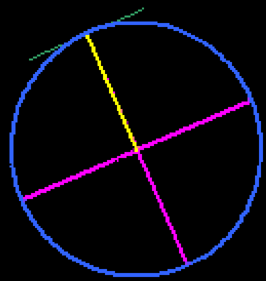
The Sun appears to move, rising on a point at the east horizon and going down on a point at the western horizon, but these points change throughout the year.

As winter approaches, the sun arcs become smaller and smaller

Apparent motion of the sun

Latitude

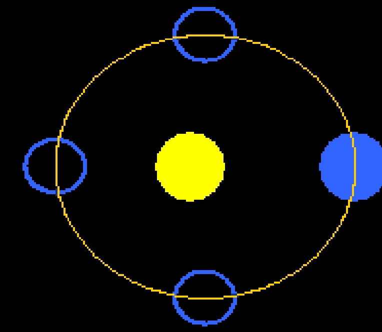
90

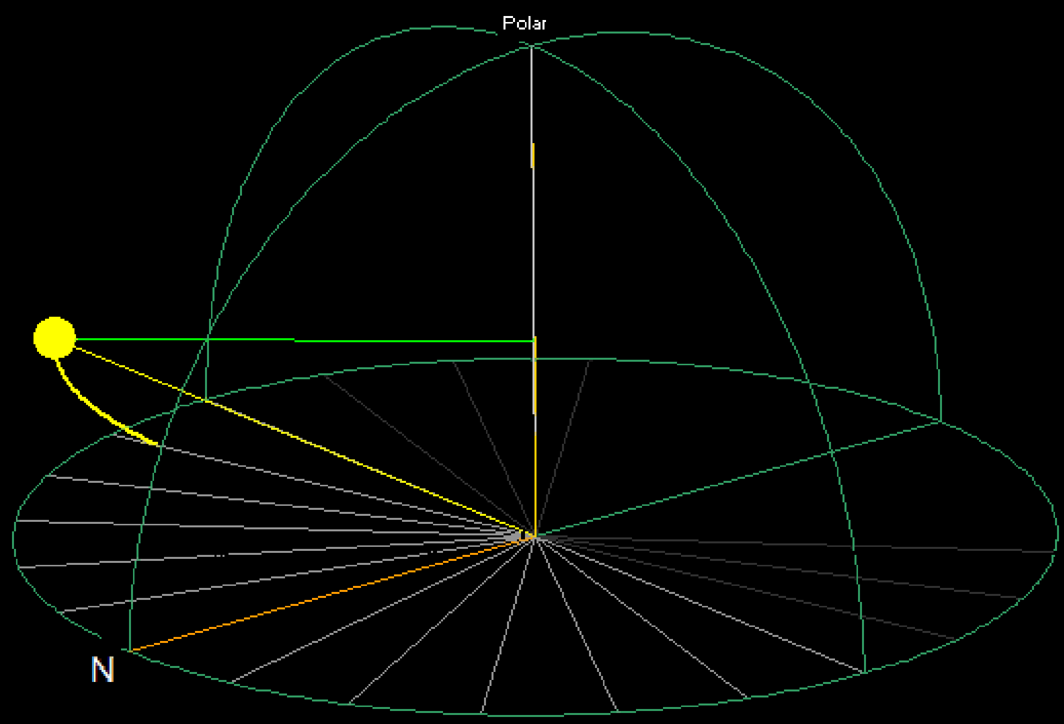


Time of year

180

Animation

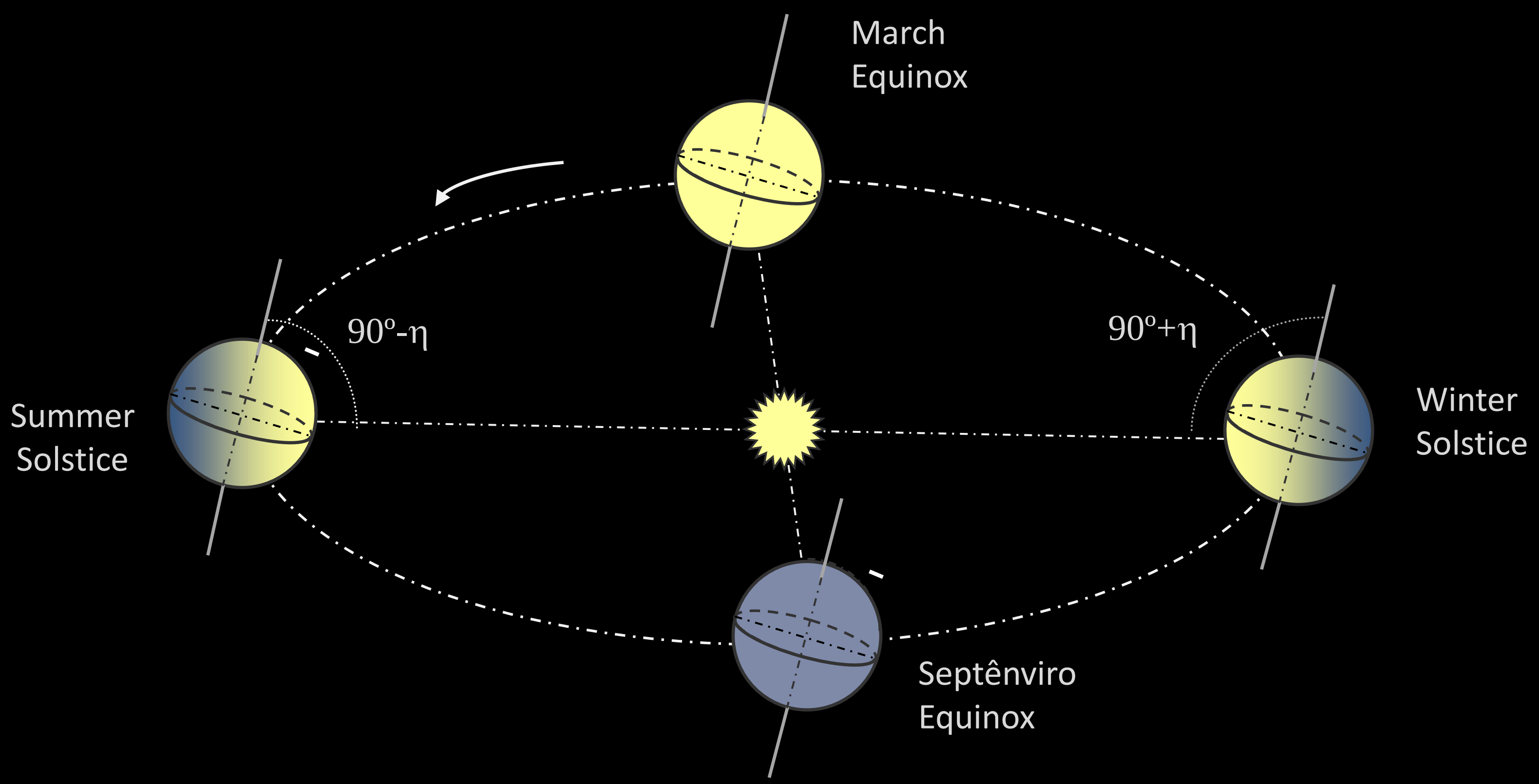




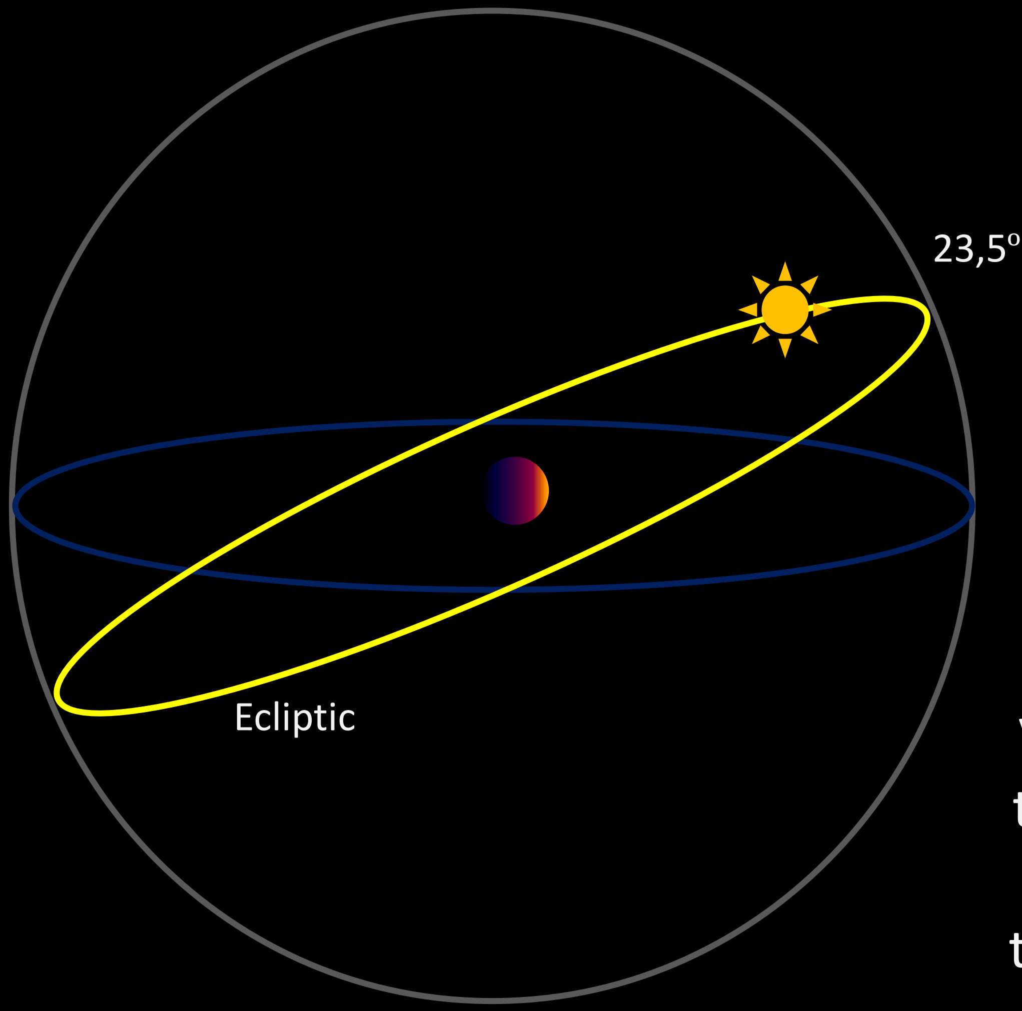
The variation of the length of the periods of light results from the translation movement of the Earth around the Sun.

In this movement the Earth keeps its axis tilted in relation to its orbital plane, which is approximately elliptical, the centre of the Sun being one of its foci.

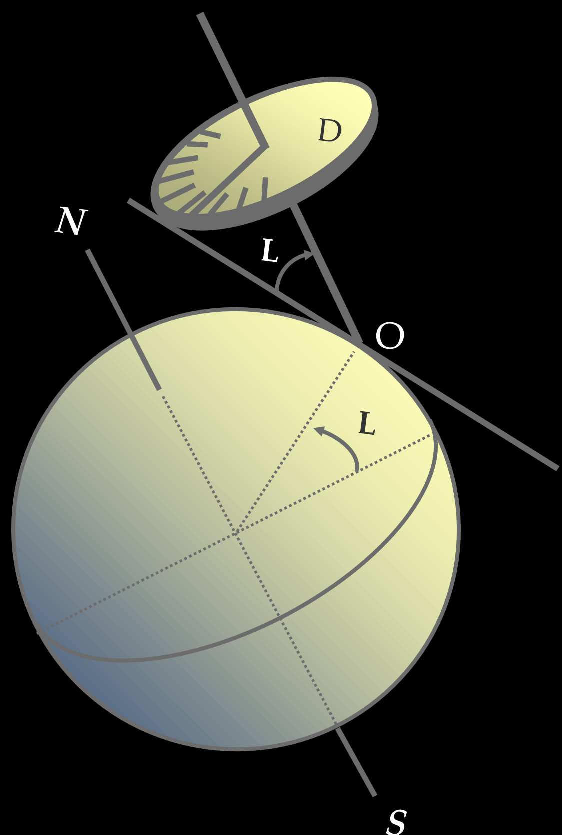
The tilt (η=23,5º) of the Earth's axis in relation to the plane of its orbit around the Sun and the elliptical shape of the orbit are what cause the height of the Sun to vary.


March
Equinox
90º-η
90º+η
Summer
Solstice
Winter
Solstice
Septênviro
Equinox

$\eta = 23{,}5^{\circ}$



Viewed from Earth, the center of the Sun is projected at a point on the celestial sphere. Throughout the year this projection describes a line, the ecliptic. The plane of the ecliptic makes with the plane of the celestial equator an angle of about 23,5 degrees.

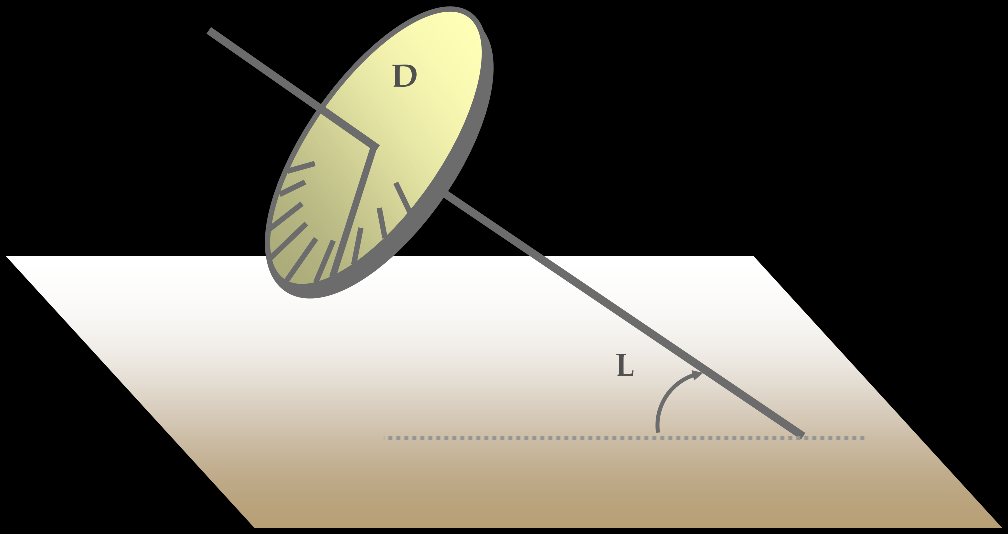


Let us imagine an instrument consisting of a disk D crossed by an axis that is perpendicular to it, placed in a local O of the Earth's surface with latitude L.

As the Sun makes its apparent motion, the shadow of the Earth's axis falls on the equatorial plane and it moves 15º per hour.

If we mark on the disk D multiple angles of 15º from the position of the shadow when the Sun passes on the local meridian (solar noon), we will obtain the marks of daylight hours.

Indeed, the instrument consisting of the disk and the axis (gnomon) reproduces the equatorial plane and the Earth's axis, and so the shadow of the axis on the disk moves 15º per hour, regardless of the season of the year.



This instrument is an equatorial sundial, since the plane containing the graduated disk is parallel to the equatorial plane.

It is possible to build variants of sundials with gnomon parallel to the Earth's axis by changing the position of the dial plane as well as its shape.

Sundials with horizontal face — horizontal dials — and sundials with vertical face — vertical dials — are very common. In these sundials, since the dial face is tilted in relation to the equatorial plane, the movement of the shadow is not uniform, being slower at around noon.

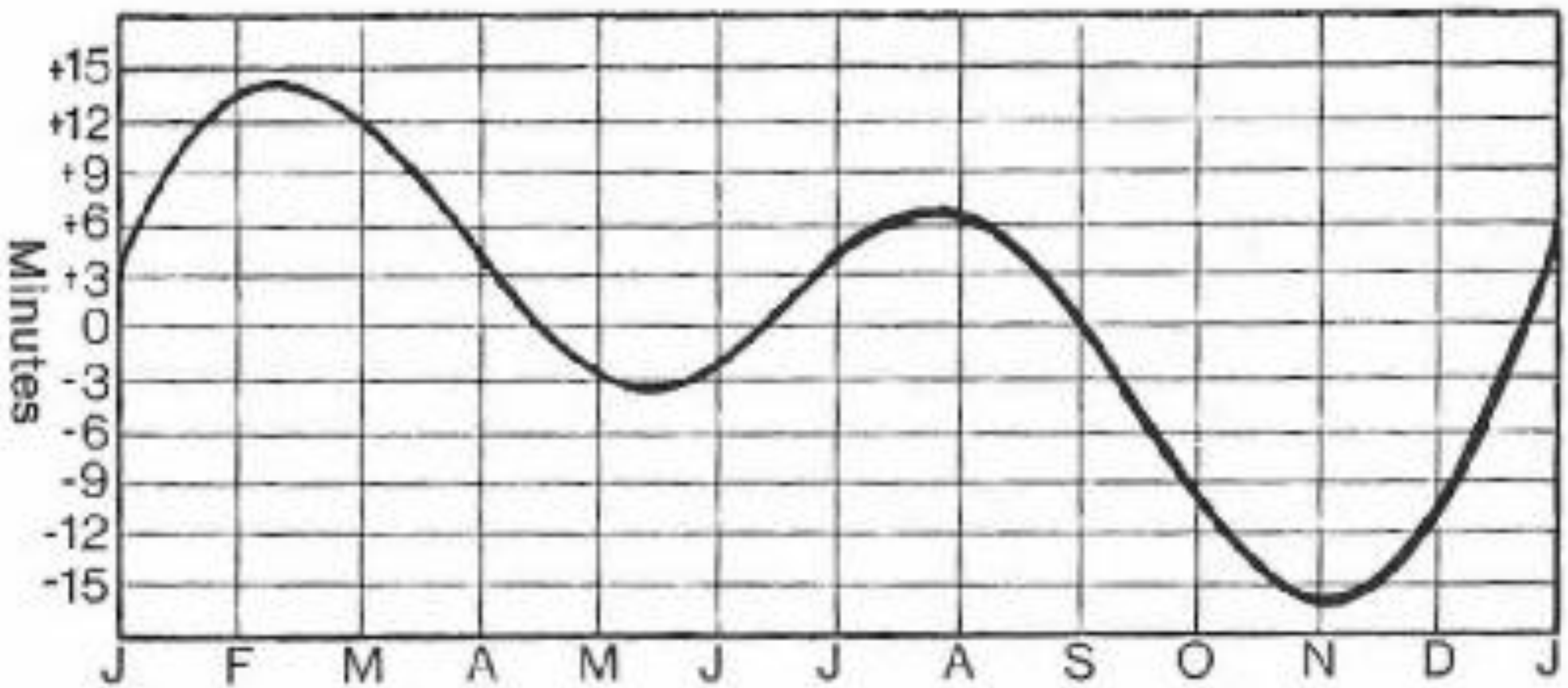
Thus, the markings do not have equal intervals: they are the projection of the equally spaced hour-lines of an equatorial dial with the same gnomon.

If we compare the hour of a sundial with the hour on our watch, we find that there are differences

This discrepancy may be quantified experimentally if, over a year, at the same time each day, we mark the end point of a vertical rod shadow. The figure obtained has the shape of an asymmetrical elongated eight illustrating the position of the Sun throughout the year and it is called analemma.

The time between two consecutive passages of the sun across the meridian of the place (solar day) only has 24 hours in 4 days of the year (16 April, 14 June, 2 September and 25 December).

The variable difference between the length of each solar day and 24 hours is usually called the ***equation of time***.

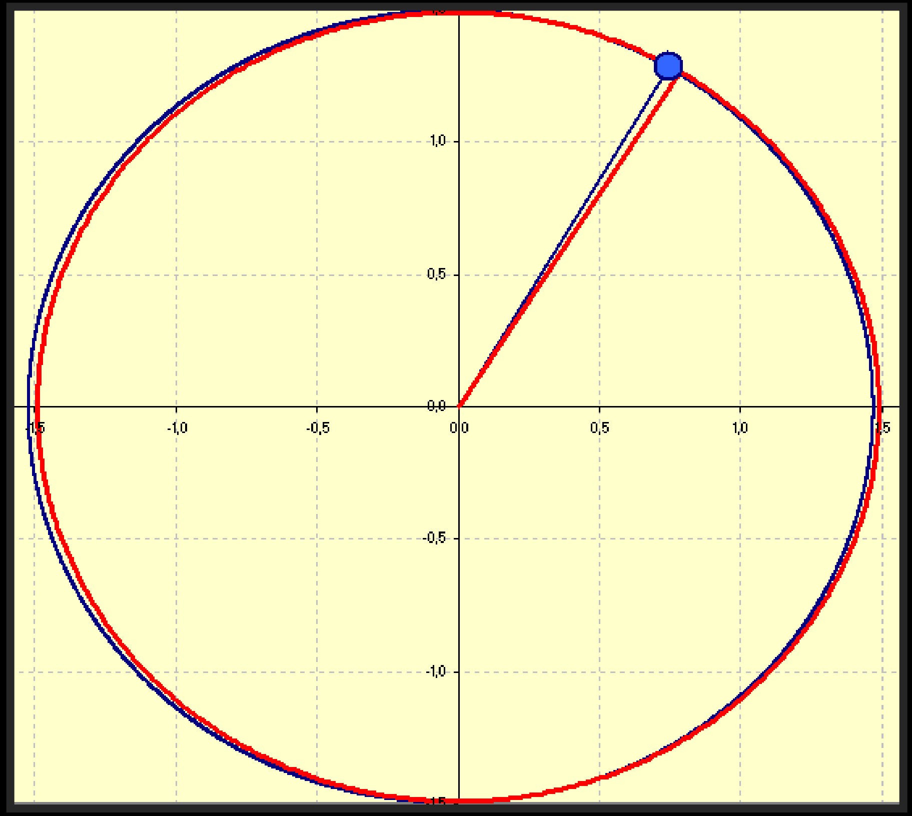


Minutes
+15
+12
+9
+6
+3
0
-3
-6
-9
-12
-15
J
F
M
A
M
J
J
A
S
O
N
D
J

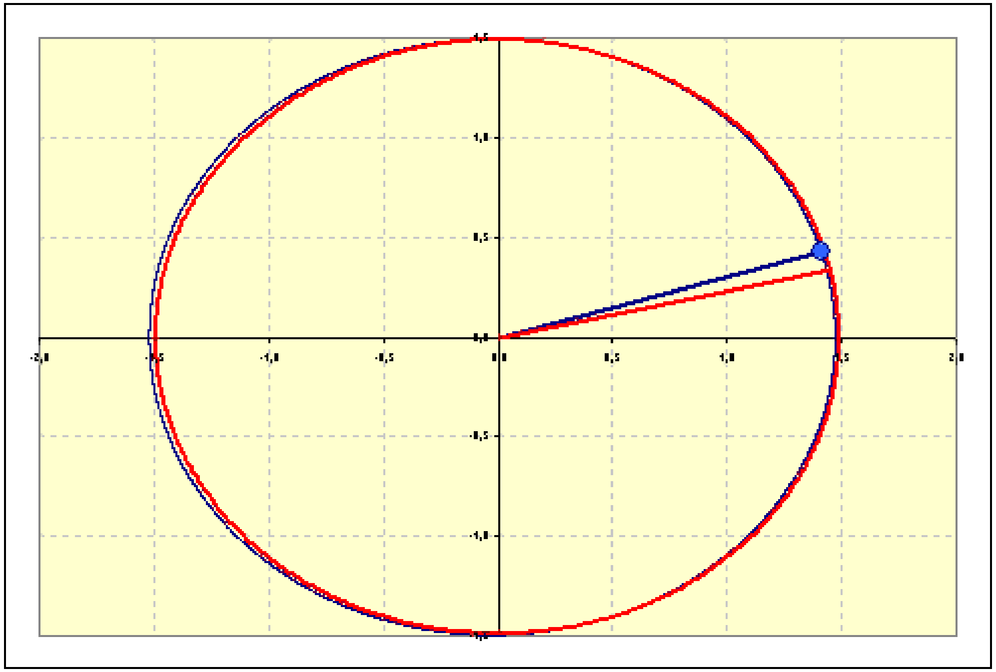
	5 th	15 th	25 th
January	+ 5m 03s	+ 9m10s	+ 12m 12s
February	+ 14m 01s	+ 14m16s	+ 13m 18s
March	+ 11m 45s	+ 9m 13s	+ 6m 16s
April	+ 2m 57s	– 0m 14s	– 1m 56s
May	– 3m 18s	– 3m 44s	– 3m 16s
June	– 1m 46s	+ 0m10s	+ 2m 20s
July	+ 4m 19s	+ 5m 46s	+ 6m 24s
August	+ 5m 59s	+ 4m 33s	+ 2m 14s
September	– 1m 05s	– 4m 32s	– 8m 04s
October	– 11m 20s	– 14m 01s	–15m 47s
November	– 16m 22s	–15m 28s	–13m 11s
December	– 9m 38s	– 5m 09s	– 0m 13s

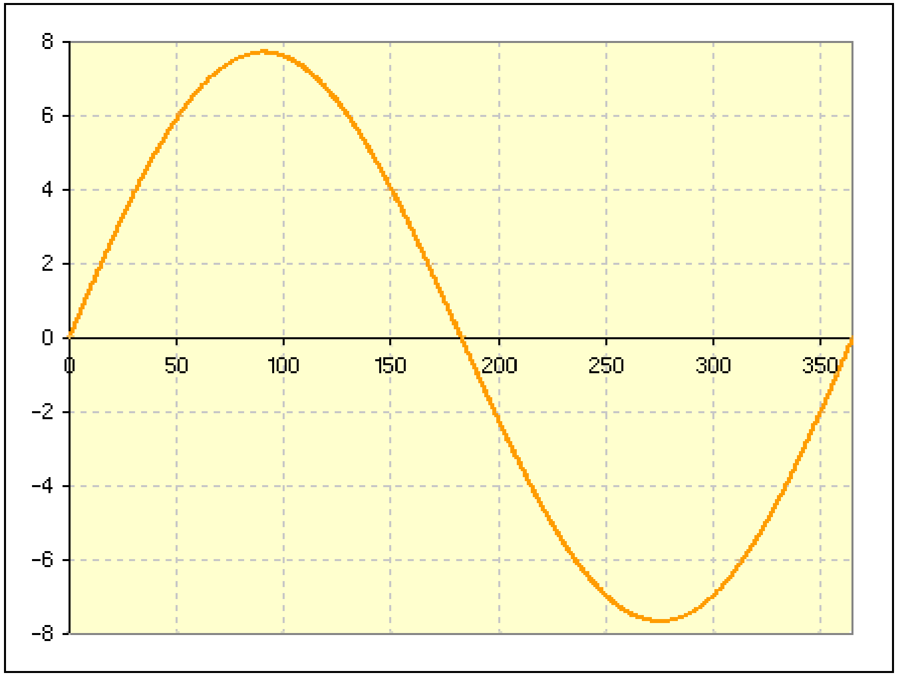
Table showing the Equation of Time on the 5th, 15th and 25th of each month

What is the influence of the shape of the Earth's orbit and of the in the difference between the length of each solar day and 24 hours ?



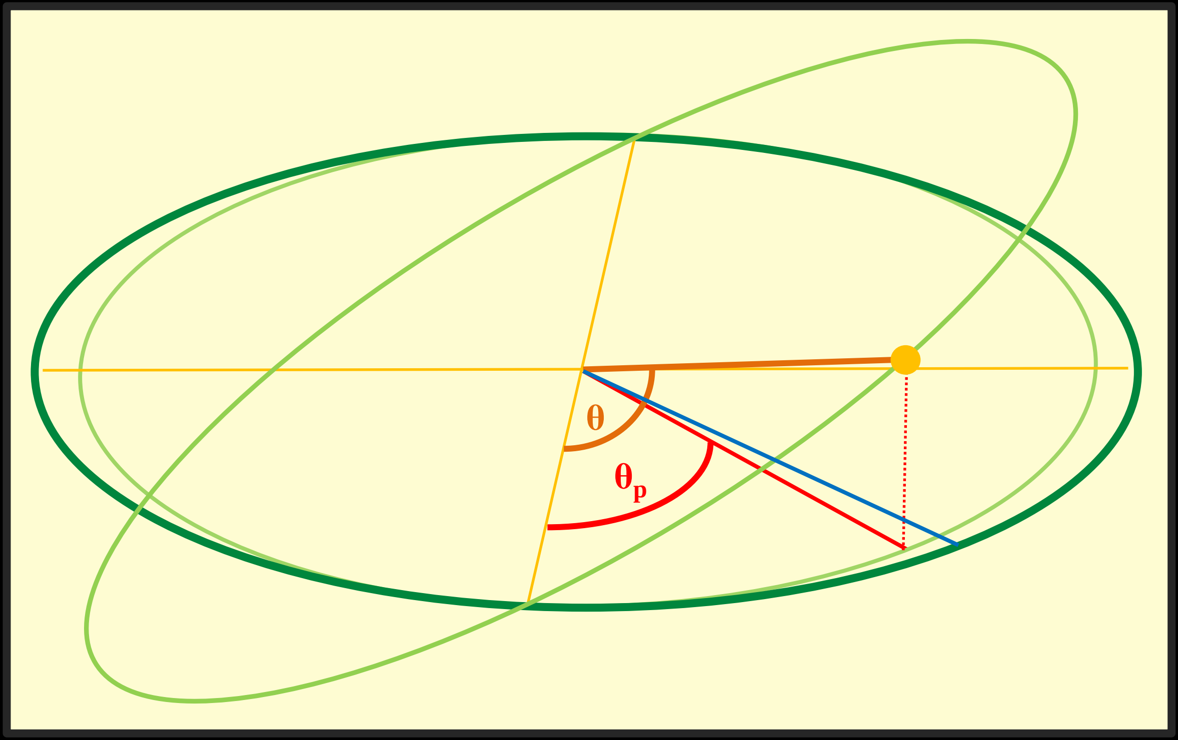
On each day of the year this difference is proportional to the difference between the angle corresponding to the position of the Earth on its trajectory and a reference angle corresponding to the uniform circular movement.





Representing these differences in a graph we obtain a sinusoid with annual period and maximum amplitude of 7,6 minutes.

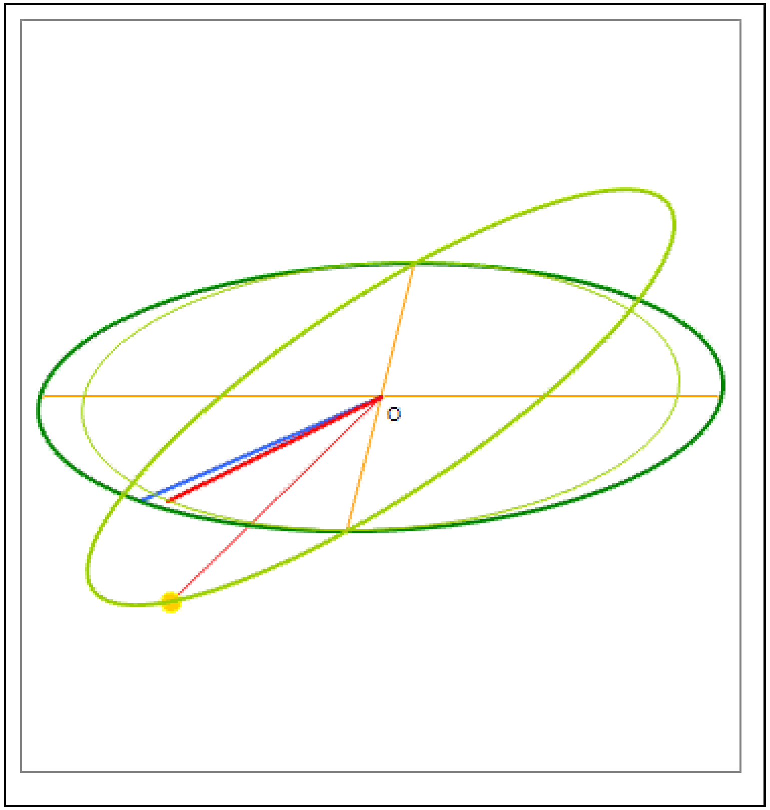
What is the influence of the obliquity of the ecliptic in the difference between the true solar time and the mean solar time?

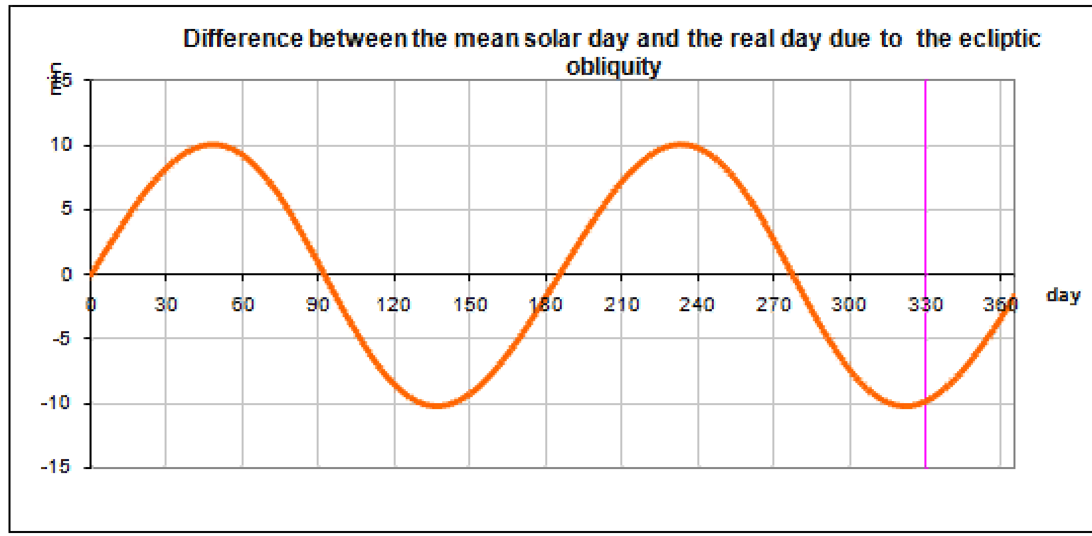


The projection of the apparent annual movement of the Sun on the plane of the celestial equator is no longer a uniform circular movement, that is, the projected angle $\boldsymbol{\theta_p}$ does not vary in a uniform way over time.

The variable difference between the length of each solar day and 24 hours depends directly from the difference (on each day of the year) between the angle $\boldsymbol{\theta_p}$ and the angle $\boldsymbol{\theta}$ corresponding to the uniform circular movement on the plane of the celestial equator.

Animação

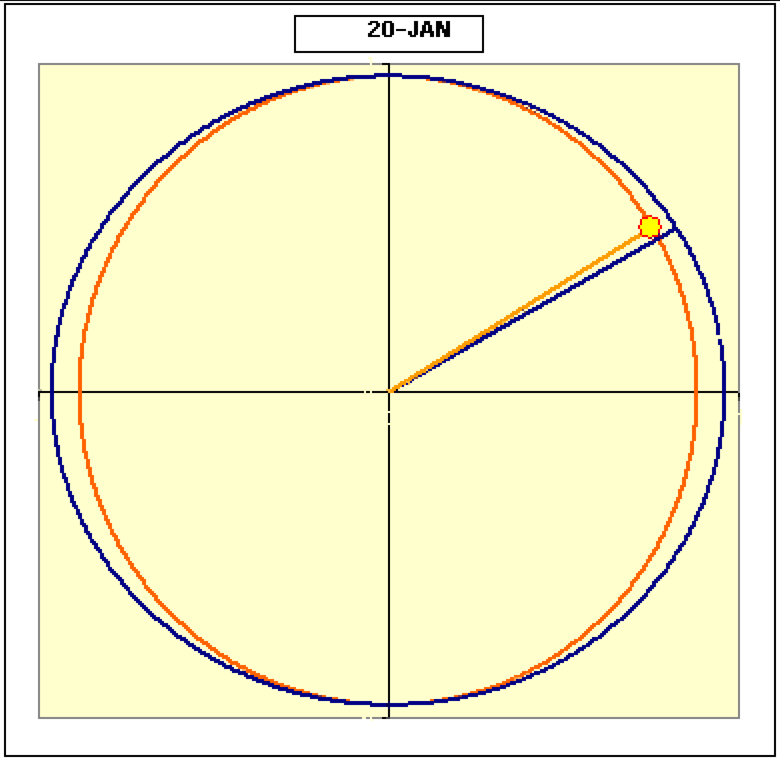


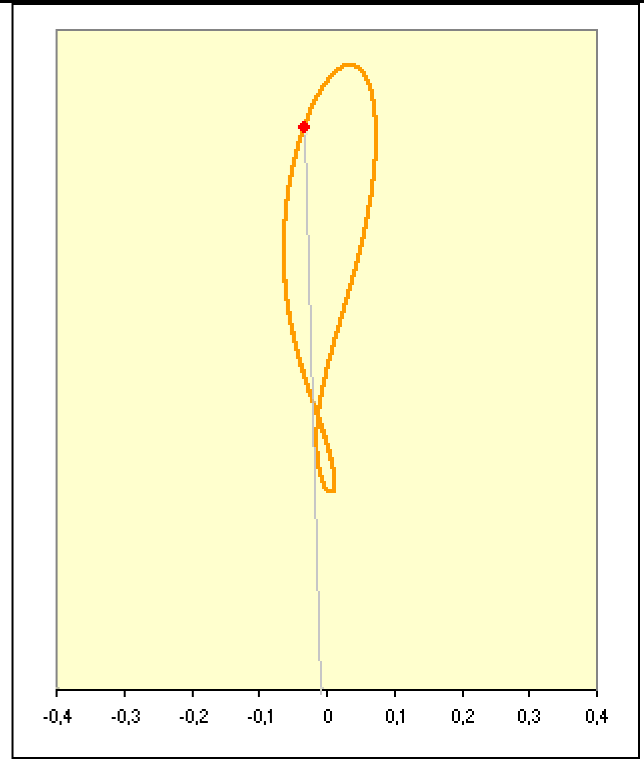


Representing these differences in a graph we obtain a sinusoid with an approximated period of 6 months .

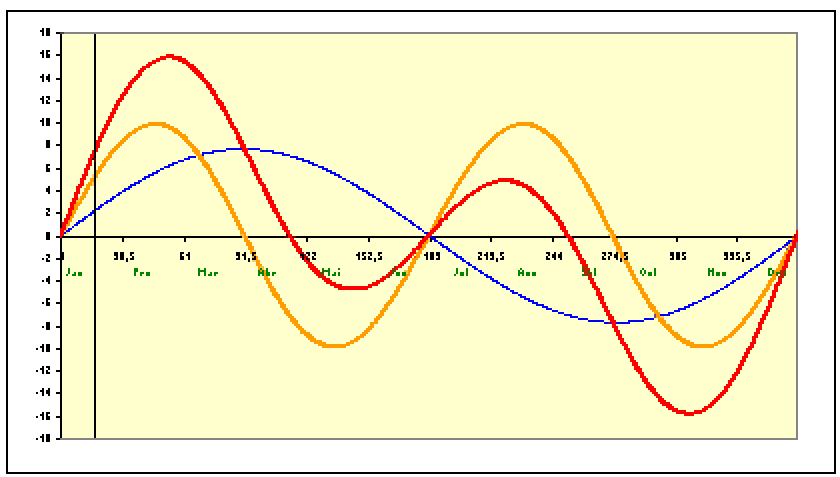
The equation of time is the sum of the two previous graphs.

The application presents a set of graphs with animation, representing the evolution of the shadow of a vertical rod at noon, as the Earth goes through its orbit, and marking the daily effects in the equation of time.





Animation



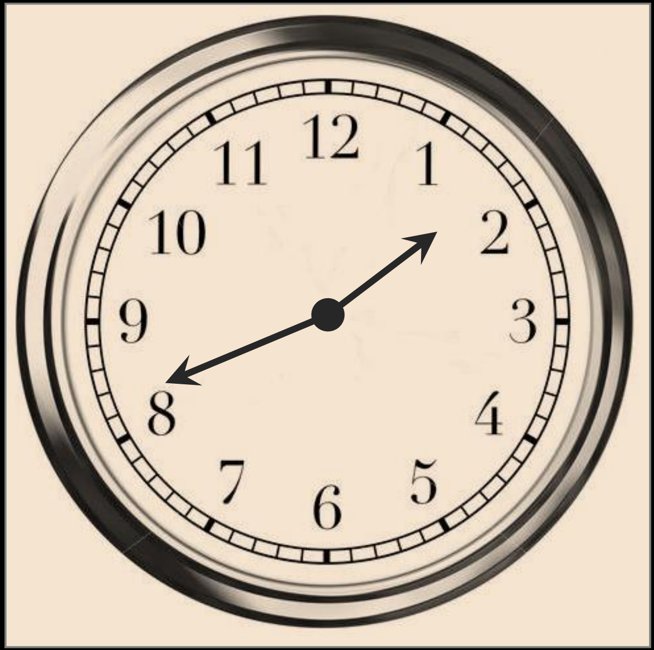
We are now able to establish the correspondence between solar time and legal time.

To obtain the correspondent legal time we must add or subtract the 4 minutes for each degree of longitude W or E.

According to the equation of time we must add 14 minutes and 1 second. Then, when the sundial indicates one hour PM, the correspondent legal time is 1h 42m 1s PM.

Longitude: 7º W

February 5, 1.00 PM (solar time)



1h 42m 1s PM (legal time)