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Determinacy versus indeterminacy

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Can a continuous function on an interval be uniquely determined if we know all the integrals of the function against the natural powers of the variable? Following Weierstrass and Stieltjes, we show that the answer is *yes* if the interval is finite, and *no* if the interval is infinite.

1 A question about integrals

Let I be a closed interval on the real line $\mathbb{R}$, say $I = [0,1], I = [0,\infty[$ or even $I = \mathbb{R}$ and suppose that $f : I \to \mathbb{R}$ is a continuous function. The question we shall address is the following:

Assume that we know that

$$\int_I f(x)x^n \, dx = 0, \quad n = 0, 1, \ldots \tag{1}$$

is true. Is it then true that $f \equiv 0$, that is, $f(x) = 0$ for all $x \in I$?

Let me immediately reveal that the answer is *yes* if I is a finite interval like $I = [0,1]$, but if the interval is infinite, like $I = [0,\infty[$ or $I = \mathbb{R}$, then the answer is *no*.

This will follow from results of two great 19th century mathematicians: Karl Weierstrass (1815–1897) for the answer *yes* and Thomas J. Stieltjes (1856–1894) for the answer *no*.

Let us first notice that the condition (1) is equivalent to the following:

$$\int_I f(x)p(x)\,dx = 0, \quad p \in \mathbb{R}[x], \tag{2}$$

where $\mathbb{R}[x]$ denotes the set of all polynomials in the variable x with real coefficients. This can be seen as follows:

On the one hand, if (2) holds, it holds in particular for the polynomials $p(x) = x^n, n = 0, 1, \ldots$, and therefore (1) holds. On the other hand, if (1) holds and $p(x) = a_n x^n + \cdots + a_1 x + a_0$ is an arbitrary real polynomial of degree n, then also (2) holds for this polynomial p because

$$\int_I f(x)p(x)\,dx = a_n \int_I f(x)x^n\,dx + \cdots + a_1 \int_I f(x)x\,dx + a_0 \int_I f(x)\,dx = 0.$$

Let us also remark that if the answer is *yes* for one finite interval, say $I = [a, b], -\infty < a < b < \infty$, then the answer is *yes* for $I = [0, 1]$ and vice versa: Note that the substitution $x = (b-a)t + a$ in the integral (2) yields

$$\int_a^b f(x)p(x)\,dx = (b-a)\int_0^1 f((b-a)t + a)p((b-a)t + a)\,dt.$$

From the above substitution we can see that if f is a continuous function on $[a, b]$, then $g(t) = f((b-a)t + a)$ is a continuous function on $[0, 1]$. And for every continuous function g on $[0, 1]$ we have that $f(x) = g((x-a)/(b-a))$ is a continuous function on $[a, b]$. This is because $x = (b-a)t + a$ maps $[0, 1]$ one-to-one onto $[a, b]$. Similarly $q(t) = p((b-a)t + a)$ is a real polynomial in t if and only if $p(x)$ is a real polynomial in x.

In 1885, Weierstrass proved what is now called the

Weierstrass approximation theorem for $[a, b]$**.** *Let f be a continuous function on $[a, b]$, where $-\infty < a < b < \infty$. Given $\varepsilon > 0$, there exists a real polynomial $p(x)$ such that*

$$|f(x) - p(x)| \le \varepsilon \quad \textit{for all } x \in [a, b]. \tag{3}$$

For an illustration of this statement, see Figure 1. Using Weierstrass's theorem we can show

Theorem 1.1. *The answer is* yes *if I is a finite interval.*

Proof. As discussed, it is enough to prove the result for $I = [0, 1]$. Let f be a continuous function on $[0, 1]$ that satisfies (1). We will show that

$$\int_0^1 |f(x)|^2\,dx = 0, \tag{4}$$

and from this we will conclude that $f \equiv 0$.

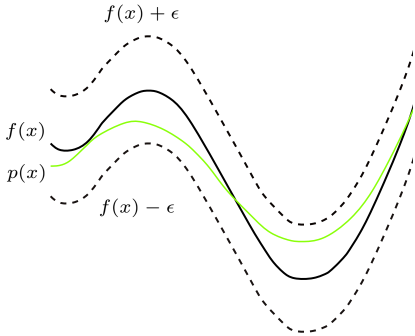

Figure 1: Given a continuous function $f(x)$ and a distance ε, we can find a polynomial $p(x)$ lying within that distance from the function $f(x)$.

To see that (4) holds, we choose an arbitrary $\varepsilon > 0$, and from Weierstrass's theorem we have a polynomial $p(x)$ such that (3) holds. We then get

$$\begin{aligned}\int_0^1 f(x)^2\,dx &= \int_0^1 f(x)(f(x)-p(x))\,dx + \int_0^1 f(x)p(x)\,dx\\ &= \int_0^1 f(x)(f(x)-p(x))\,dx,\end{aligned}$$

where for the last equality we used the assumptions (2). From this we get (remember that $f(x)^2$ is never negative)

$$\begin{aligned}0 \le \int_0^1 f(x)^2\,dx = &\left|\int_0^1 f(x)(f(x)-p(x))\,dx\right|\\ &\le \int_0^1 |f(x)|\,|f(x)-p(x)|\,dx \le \varepsilon \int_0^1 |f(x)|\,dx,\end{aligned}$$

and since $\int_0^1 |f(x)|\,dx$ is independent of $\varepsilon > 0$, we get

$$\int_0^1 |f(x)|^2\,dx = \int_0^1 f(x)^2\,dx = 0$$

as claimed in (4).

Now let us proceed to show that $f \equiv 0$. If there is some $x_0 \in [0,1]$ for which $f(x_0) \neq 0$ then $|f(x_0)| = c > 0$. Since f is continuous, $|f(x)| > c/2$ for all x in some small interval around x_0, which will contribute a nonzero amount to $\int_0^1 |f(x)|^2\,dx$ and this will contradict (4). $\square$

Theorem 1.2 (Stieltjes [8]). *The answer is* no *if I is an infinite interval.*

Proof. We will prove the theorem by giving explicit examples of functions that satisfy (1), yet are not everywhere zero. We will cover all possible forms of infinite intervals. Let us first assume that $I = [0, \infty[$. Define the continuous function f on $]0, \infty[$ by

$$f(x) = \exp\Big(-\frac{1}{2}(\log x)^2\Big)\sin(2\pi \log x), \quad x > 0. \tag{5}$$

Since the sine function is bounded (between 1 and -1), and $\log x$ tends to $-\infty$ as $x \to 0$, it follows that $\lim_{x\to 0} f(x) = 0$. Thus, defining $f(0) = 0$ we now have that f is continuous on $[0, \infty[$. Since $\exp(-\frac{1}{2}(\log x)^2)$ is never zero and $\sin(2\pi \log x)$ is zero only when $\log x = \frac{k}{2}$ for some integer k, there are infinitely many points $x \in [0, \infty[$ where $f(x) \neq 0$.

Nevertheless, we can follow Stieltjes in showing that f is a function that satisfies (1). Making the substitution $u = \log x$ in the integral, we get

$$\begin{aligned}\int_I f(x)x^n\,dx &= \int_{-\infty}^{\infty} \exp\Big(-\frac{1}{2}u^2 + u(n+1)\Big)\sin(2\pi u)\,du\\ &= \exp\Big(\frac{1}{2}(n+1)^2\Big)\int_{-\infty}^{\infty}\exp\Big(-\frac{1}{2}(u-(n+1))^2\Big)\sin(2\pi u)\,du\\ &= \exp\Big(\frac{1}{2}(n+1)^2\Big)\int_{-\infty}^{\infty}\exp\Big(-\frac{1}{2}v^2\Big)\sin(2\pi(v+n+1))\,dv,\end{aligned}$$

where for the last equality we made the substitution $v = u - (n+1)$. Using the periodicity of the sine function, we finally get

$$\int_I f(x)x^n\,dx = \exp\Big(\frac{1}{2}(n+1)^2\Big)\int_{-\infty}^{\infty}\exp\Big(-\frac{1}{2}v^2\Big)\sin(2\pi v)\,dv = 0,$$

because the integral of an odd function is 0.

The function $f(x-a)$ will provide the result for infinite intervals of the form $[a, \infty[$. Similarly, the function $f(a-x)$ will take care of the infinite interval $]-\infty, a]$.

Finally, we claim that the function $|x|f(x^2)$ is an example for $I = \mathbb{R}$. We shall show that

$$\int_{-\infty}^{\infty} |x|f(x^2)x^n\,dx = 0, \quad n = 0, 1, \ldots.$$

This is certainly true when n is odd, because the integral of an odd function is zero. When n is even, say $n = 2k$, we get

$$\int_{-\infty}^{\infty} |x|f(x^2)x^{2k}\,dx = 2\int_0^{\infty} xf(x^2)x^{2k}\,dx = \int_0^{\infty} f(t)t^k\,dt = 0,$$

where we have used the substitution $t = x^2$. $\square$

2 The Stone–Weierstrass Theorem

There exist many proofs of Weierstrass's theorem. In the case of $I = [0, 1]$ we mention the method due to S. Bernstein (1880–1968). The nth Bernstein polynomial of a continuous function f on $[0, 1]$ is defined as

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{k}{n} x^k (1-x)^{n-k}, \tag{6}$$

and Bernstein proved that given $\varepsilon > 0$, there exists an integer N_0 such that

$$|f(x) - B_n(f)(x)| \leq \varepsilon \quad \text{for all } x \in [0, 1] \text{ and all } n \geq N_0.$$

This is, indeed, a remarkable work. Not only did Bernstein prove Weierstrass's theorem, he also provided an explicit formula for the polynomials we can use to approximate f. For one example demonstrating the usefulness of Weierstrass's theorem and Bernstein's proof, think of modern computers. With all of their power and utility, in their core, they can perform only the simplest arithmetic operations: addition, subtraction, multiplication and a few more. Yet everyday use of computers requires the evaluation of complicated functions; for example, drawing a character in a video game relative to the position of the player, performing a sophisticated search in a large database, analyzing the frequencies of recorded sound, and so on. Polynomials are easy for computers: they are a combination of addition, multiplication and exponentiation. Bernstein's proof gives an explicit polynomial we can use to approximate any continuous function within any error margin.[1]

A far-reaching and important generalization of Weierstrass's theorem was obtained by Marshall Stone (1903–1989) in 1937. He replaced the finite interval $[a, b]$ by a *compact Hausdorff space* X, a concept from topology generalizing closed and bounded subsets of the k-dimensional space $\mathbb{R}^k$. Topology is one of the really important new subjects of 20th century mathematics, see [6].

Stone–Weierstrass theorem. *Let X be a compact Hausdorff space and let $\mathcal{A}$ be a subset of the set $C(X)$ of continuous real-valued functions on X with the following properties:*

(i) *$\mathcal{A}$ is an algebra, that is, if $f, g \in \mathcal{A}$ and $\lambda \in \mathbb{R}$, then λf, $f + g$, and $f \cdot g \in \mathcal{A}$.*
(ii) *The constant functions belong to $\mathcal{A}$ and $\mathcal{A}$ separates the points of X, that is, to any $x_1, x_2 \in X$, $x_1 \neq x_2$, there exists $p \in \mathcal{A}$ such that $p(x_1) \neq p(x_2)$.*

Then for any function $f \in C(X)$ and $\varepsilon > 0$ there exists $p \in \mathcal{A}$ such that

$$|f(x) - p(x)| \leq \varepsilon \quad \textit{for all } x \in X.$$

[1] These days, some real-world applications use polynomials that are more efficient than the original Bernstein polynomials.

The set $\mathcal{A} = \mathbb{R}[x]$ of real polynomials has the properties (i) and (ii) as subset of $C([a,b])$, which makes Weierstrass's theorem true as a special case.

3 Moment problems

Stieltjes finished an important memoir [8] a few months before his premature death in 1894. In this work he defined and solved what is known as the Stieltjes moment problem:

> Given a sequence of real numbers $s_0, s_1, \ldots$, find necessary and sufficient conditions such that (s_n) are the *moments* of a positive measure μ on the interval $[0, \infty[$, that is,
>
> $$s_n = \int_0^\infty x^n \, d\mu(x), \quad n = 0, 1, \ldots. \tag{7}$$

Simply put, a *measure* assigns "length" or "weight" to subsets of $[0, \infty[$; so it is usually non-negative. A familiar example would be assigning to a segment $[a, b] \in \mathbb{R}$ its length $b - a$. Another example would be that of a probability measure which assigns to each set the probability of a sample to be within that set ("weight" of the set). This is usually achieved by integrating an appropriate *probability density function* over the set.

Now it can be seen that the moment problem takes a very similar form to that of (1); only replacing the requirement of the integral being equal to zero with the integral being equal to a sequence of prescribed values. A measure can have a continuous density like e^{-x}, and in this case (7) translates to

$$s_n = \int_0^\infty x^n e^{-x} \, dx = n!, \quad n = 0, 1, \ldots,$$

so the sequence $(n!) = 1, 1, 2, 6, 24, \ldots$ is an example of a Stieltjes moment sequence. If the measure is so-called discrete, that is, with weights $c_k > 0$ in the points $0 \le x_1 < x_2 < \cdots$, then the moments are

$$s_n = \sum_{k=1}^\infty c_k x_k^n, \quad n = 0, 1, \ldots.$$

As an example, if $x_k = k, c_k = \exp(-k), k = 1, 2, \ldots,$[2] then

$$s_n = \sum_{k=1}^\infty k^n \exp(-k) = (-1)^n f^{(n)}(1), \quad \text{where } f(x) = (\exp(x) - 1)^{-1}, \ x > 0,$$

[2] Recall that $\exp(x) = e^x$, where e is the base of the natural logarithm.

because

$$f(x) = \frac{\exp(-x)}{1-\exp(-x)} = \sum_{k=1}^{\infty} \exp(-kx), \quad x > 0.$$

There are more types of measures than the above examples show, but we will not discuss this further.

Stieltjes also realized that some moment sequences (s_n) are *determinate* but others are *indeterminate*. By this he meant that in the determinate case, there is exactly one measure satisfying (7), and in the indeterminate case there can be more than one solution and then automatically infinitely many measures μ satisfying (7).

If the measure μ has the density

$$d(x) = \exp\Big(-\frac{1}{2}(\log x)^2\Big),$$

the moments are

$$s_n = \int_0^{\infty} d(x)x^n\,dx = \sqrt{2\pi}\exp\Big(\frac{1}{2}(n+1)^2\Big), \quad n = 0, 1, \ldots.$$

To see this, make the substitution $u = \log x$, use the trick from the proof of Theorem 1.2 and finally use the Gaussian integral

$$\sqrt{2\pi} = \int_{-\infty}^{\infty} \exp\Big(-\frac{1}{2}u^2\Big)\,du.$$

This means that $d(x)/\sqrt{2\pi e}$ is a probability density function over the interval $[0,\infty[$, called a *log-normal density*.

Stieltjes' result about the function (5) means that

$$s_n = \int_0^{\infty} d(x)(1 + c\sin(2\pi\log x))x^n\,dx = \sqrt{2\pi}\exp\Big(\frac{1}{2}(n+1)^2\Big)$$

for all $c \in \mathbb{R}$. For the infinitely many values of $c \in [-1, 1]$, the expression

$$d_c(x) = d(x)(1 + c\sin(2\pi\log x))$$

is non-negative for $x \geq 0$, and therefore defines a positive measure.

Stieltjes had frequent correspondence with his thesis advisor and mentor Charles Hermite (1822–1901), see [2], which contains the following remark in a letter from Stieltjes to Hermite dated January 30, 1892:

L'existence de ces fonctions $f(x)$ qui, sans être nulles, sont telles que

$$\int_0^\infty f(x)x^n\,dx = 0, \quad n = 0, 1, \dots,$$

me paraît très remarquable.

The existence of these functions $f(x)$ that, without being zero, are such that

$$\int_0^\infty f(x)x^n\,dx = 0, \quad n = 0, 1, \dots,$$

appears very remarkable to me.

Hermite

Stieltjes

Moment sequences play an important role in probability theory and statistics, where quantities like *mean, variance, skewness* and *kurtosis* can be expressed in terms of the moment sequence (s_n). For a probability distribution (a positive measure) μ with moments (7) we have $s_0 = 1$, s_1 is called the *mean* and $s_2 - s_1^2$ is called the *variance*. Most of the classical probability distributions like the exponential, normal and Poisson distribution have determinate moment sequences, and they are thus uniquely determined by their moments. Probability distributions with indeterminate moment sequences were considered a bit exotic, when they were discovered by Stieltjes. During the last 25 years a number of new indeterminate moment sequences have been discovered. One knows that the sequence $(n!)^c$, $n = 0, 1, \dots$ is determinate for $0 < c \le 2$ but indeterminate for $c > 2$, cf. [3]. The moment problem is treated in a wonderful book by Akhiezer [1] from 1965 and in a very recent monograph [7]. There is renewed interest in indeterminate moment problems, see [4], and in multidimensional moment problems related to algebraic geometry, see [5].

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