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# Geometry behind one of the Painlevé III differential equations

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The Painlevé equations are second order differential equations, which were first studied more than 100 years ago. Nowadays they arise in many areas in mathematics and mathematical physics. This snapshot discusses the solutions of one of the Painlevé equations and presents old results on the asymptotics at two singular points and new results on the global behavior.

## 1 The Painlevé equations

Asking for functions $f\colon \mathbb{R} \to \mathbb{R}$, probably first the polynomials in $\mathbb{R}[x]$ like $f(x) = x^2 + x - 1$ come into mind. The next examples might be quotients of polynomials like $g(x) = \frac{x^3-2x-1}{x^2+1}$. However, many important functions cannot be defined by a polynomial relation. These are the so-called *transcendental functions*, such as the sine function, cosine, logarithms, and so on.

The most prominent one of the transcendental functions is the exponential function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = e^x$. It can be characterized as follows: It is the unique solution of the first order differential equation $f'(x) = f(x)$ with the initial value $f(0) = 1$.

At the end of the 19th century, the exponential function and a number of other transcendental functions were well known, and their importance in many applications was obvious. They can be called *classical transcendental functions*.

The French mathematician Paul Painlevé (1863–1933) then undertook a search for *new* transcendental functions. To find them he looked at *second* order differential equations and their solutions. His work [10] was continued by Bertrand Gambier (1879–1954) in [2]. They obtained a list of 50 families of second order differential equations which are distinguished (within all second order differential equations) by the following property, which is nowadays called the *Painlevé property*:

**Painlevé property.** *In* $\mathbb{C}$ *or* $\mathbb{C} - \{0\}$ *or* $\mathbb{C} - \{0; 1\}$ *(depending on the differential equation) any local solution (which is defined near a point* $x_0$*) extends near any path (which starts at* $x_0$*) to a function which is almost complex analytic, the bad points being only poles.*

Recall that a function is *real analytic* in a point $x_0$ if it can be written locally around $x_0$ (such that it is defined near the point $x_0$) in the form of a convergent power series $\sum_{k=0}^{\infty} a_k(x - x_0)^k$. That means that the value at a point $x$ near $x_0$ of such a function is the limit for $n \to \infty$ of the sequence of numbers which are given by the finite sums $\sum_{k=0}^{n} a_k(x - x_0)^k$. Functions that are real analytic in any point are called *real analytic functions*. This notion extends to complex valued functions in an open subset of the set $\mathbb{C}$ of complex numbers. Then those functions are called *complex analytic* or *holomorphic*.

A *pole* is a point $x_0$ such that near $x_0$ the function can be written as a convergent series $\sum_{k=p}^{\infty} a_k(x - x_0)^k$ for some $p \in \mathbb{Z}_{<0}$ and some $a_p \neq 0$. So in contrast to before, we allow finitely many terms of the form $a_{-k}\frac{1}{(x-x_0)^k}$ in the sum. The number $p$ is the *order* of the pole. A pole of order 1 is called a *simple pole*.

The Painlevé property contains two non-trivial properties, first that solutions extend arbitrarily far, that is, along arbitrary paths, and second, that they have at most poles and no essential singularities.

Painlevé listed correctly all functions that satisfy the Painlevé property but his proof that they fulfill this property turned out to be incomplete. However, nowadays there are several proofs, see for example [3]. Also the property that most (not all) of the solutions of the Painlevé equations are new transcendental functions, was made precise and was proved only much later (by Nishioka and Umemura and others in the 1980s).

It turned out that the list of the 50 families can be simplified to a list of six out of the 50 families, which are called the *Painlevé equations of types I, II, III, IV, V and VI.* The other families can be reduced either to one of the equations in these six families, to other well-understood differential equations, or to classical transcendental functions. The families of types I, II, III, IV, V and VI have 0, 1, 2, 2, 3, and 4 parameters, respectively. We will see what the two parameters are for the Painlevé III equations in the next section.

In the second half of the 20th century, physicists became interested in the Painlevé equations because many applications in mathematical physics emerged. Mathematicians became intrigued by the Painlevé equations because of relations to many areas in mathematics. Some expositions with lots of references are [1, 3, 6, 11].

## 2 One of the Painlevé III equations

In this snapshot, one of the Painlevé III equations will be considered in detail. It is the equation

$$f''(x) = \frac{(f'(x))^2}{f(x)} - \frac{1}{x}f'(x) + 4(f(x))^3 - 4\frac{1}{f(x)}. \tag{1}$$

Because $\frac{1}{x}$ appears, the equation is defined on $\mathbb{C} - \{0\}$. Though in this snapshot, we will look only at real valued solutions on $\mathbb{R}_{>0}$. Of all the Painlevé III equations, this is the most important one in mathematical physics, and in this application the real solutions on $\mathbb{R}_{>0}$ are used. And it is the one with the most symmetries. The symmetries will be explained in this snapshot.

Equation (1) does not look appealing at all. A prettier equation is obtained if one chooses locally a function $g(x)$ with $f(x) = e^{g(x)/2}$. Then a short calculation shows that $g(x)$ satisfies the differential equation

$$(x\partial_x)^2 g(x) = 16x^2 \cdot \sinh(g(x)) \quad \Big[ = \; 8x^2(e^{g(x)} - e^{-g(x)}) \Big]. \tag{2}$$

Here we use $\partial_x = d/dx$. By looking at equation (2) one finds: If $g$ is a solution of equation (2), then also $g + 2\pi i, -g$, and $-g + 2\pi i$ are solutions of equation (2). By plugging these three terms into $f(x) = e^{g(x)/2}$, we see that if $f$ is a solution of equation (1), then also $-f$, $f^{-1} = \frac{1}{f}$, and $-f^{-1} = -\frac{1}{f}$ are solutions of equation (1). The Painlevé property above means the following for real solutions on an open subset of $\mathbb{R}_{>0}$: If a solution tends to infinity approaching one point $x_0$ from the left or the right, then it has a pole of some order $p$ at this point. This implies especially that the solution extends uniquely to a function on the other side of $x_0$. More generally, any solution on an open subset of $\mathbb{R}_{>0}$ extends to a function on all of $\mathbb{R}_{>0}$ which might have poles at some points and which is everywhere else real analytic.

If $f$ has a pole at $x_0$ then $f^{-1} = \frac{1}{f}$ is also a solution of equation (1), and $f^{-1}$ has a zero at $x_0$. Let $\sum_{k=1}^{\infty} a_k(x - x_0)^k$ be the power series representation of $f^{-1}$ at a zero $x_0$. Note that we do not need a summand of the form $a_0(x - x_0)^0$ if $x_0$ is a zero. One can plug this power series into the equation (1), using $(f^{-1})'(x) = \sum_{k=1}^{\infty} ka_k(x-x_0)^{k-1}$ and $(f^{-1})''(x) = \sum_{k=2}^{\infty} k(k-1)a_k(x-x_0)^{k-2}$, and compare the coefficients of the different powers of $(x - x_0)$ on both sides of the equation. With a bit of computation, one finds the following:

$$\begin{aligned} &a_1 = \pm 2, \quad a_2 = \frac{2a_1}{x_0}, \quad a_3 \text{ is arbitrary in } \mathbb{R}, \\ &a_k \text{ for } k \geq 4 \text{ are determined by } a_1 \text{ and } a_3. \end{aligned} \tag{3}$$

Therefore we always have $a_1 \neq 0$ and so the zero of $f^{-1}$ at $x_0$ is a simple zero and the pole of $f$ at $x_0$ is a simple pole.

Therefore all solutions of equation (1) have at most *simple* zeros and/or *simple* poles, and both of them come in two types: A simple zero $x_0$ of a solution $f$ is called *of type* $[0+]$ if $f'(x_0) = 2$ which corresponds to $a_1 = 2$. It is called *of type* $[0-]$ if $f'(x_0) = -2$ which corresponds to $a_1 = -2$. A simple pole $x_0$ of $f$ is called *of type* $[\infty+]$ respectively $[\infty-]$, if $x_0$ is a simple zero of $f^{-1}$ of type $[0+]$ respectively $[0-]$. The zeros and poles have to be considered on the same footing and have to be treated with equal care. In fact, already the appearance of $\frac{1}{f}$ in equation (1) indicates that the zeros of solutions have also to be treated with special care, not only the poles of solutions. Points $x_0 \in \mathbb{R}_{>0}$ at which a solution has a simple zero or pole are called *singular*, the other points are called *regular*.

Consider a point $x_0 \in \mathbb{R}_{>0}$. We denote by $\mathbb{R}^*$ the set $\mathbb{R} - \{0\}$. Then for any two numbers $a_0 \in \mathbb{R}^*$ and $a_1 \in \mathbb{R}$, there is a unique solution $f$ of equation (1) with $f(x_0) = a_0$ and $f'(x_0) = a_1$. Both, uniqueness and local existence, are elementary facts from the theory of ordinary differential equations. The global existence follows with the Painlevé property. For such a solution $f$, $x_0$ is a regular point. The values $(a_0, a_1) \in \mathbb{R}^* \times \mathbb{R}$ are called *initial values* of the solution $f$. Thus, for any initial values and any $x_0 \in \mathbb{R}_{>0}$, there is a unique solution of equation (1) with these initial values at $x_0$.

A variant of this holds also at singular points: For any pair $(a_1, a_3) \in \{\pm 2\} \times \mathbb{R}$, there is a unique solution $f$ of equation (1) with a simple zero at $x_0$ and with power series representation at $x_0$ with coefficients as in equation (3). This fact builds on equation (3). The pair $(a_1, a_3) \in \{\pm 2\} \times \mathbb{R}$ gives the initial values of a solution at a zero of it. By passing to $f^{-1}$, one can interpret $(a_1, a_3)$ also as the initial values of a solution at a pole of it. As before, uniqueness and local existence follow from the theory of ordinary differential equations, and the global existence follows with the Painlevé property.

We now see that all solutions to equation (1) can be constructed by choosing two parameters, as indicated at the end of Section 1.

It turns out that one can glue the sets $\mathbb{R}^* \times \mathbb{R}$ of initial values for regular points and the sets $\{\pm 2\} \times \mathbb{R}$ and $\{\pm 2\} \times \mathbb{R}$ of initial values for zeros and poles to one set $M_{ini}(x_0)$. Here "gluing" means that the set $M_{ini}(x_0)$ is a geometric object such that each point of it corresponds to one of the possible initial values.

In fact, $M_{ini}(x_0)$ is a "real algebraic surface" and can be described by four open pieces (four *charts*) glued along $\mathbb{R}^* \times \mathbb{R}$. Each of the charts is isomorphic to $\mathbb{R} \times \mathbb{R}$ (that means, there is a $1-1$ correspondence) and contains the set of regular initial values and one of the four components (isomorphic to $\mathbb{R}$) of the set of singular initial values. The real algebraic surface $M_{ini}(x_0)$ is constructed in [4, Chapter 1]. Spaces of initial values for most Painlevé equations had been constructed in [9].

Denote by $M_{sol}$ the set of all real valued solutions of equation (1) on $\mathbb{R}_{>0}$ (poles are allowed).

Then there is a natural bijection, that is, a $1-1$ correspondence, between $M_{ini}(x_0)$ and $M_{sol}$ which sends any initial values to the corresponding unique solution. As this is a bijection for every chosen $x_0$, also all the spaces $M_{ini}(x_0)$ look the same for different $x_0$. The next section explains how the sets look like roughly.

## 3 Asymptotics near 0 and $\infty$

The set $M_{sol}$ is given in a very abstract way so it is hard to work with it. As we have said before, within the sets $M_{ini}(x_0)$, there is no best one. But it turns out that there is a best possible reference set, a surface which is in a natural bijection to $M_{sol}$ and which can be described concretely. We will construct it as a set of initial values at the limit point $x = 0$. These initial values will describe the asymptotics at $x = 0$ of solutions of equation (1).

A more conceptual way would be to construct it as "monodromy data" of certain associated "meromorphic connections" [5, 1, 12, 4], but explaining that is beyond the scope of this snapshot.

It is a major question for many differential equations how the solutions look like *asymptotically* (= roughly) at certain singular points. In our case, the singular points are 0 and $\infty$. The asymptotics of the solutions at 0 and $\infty$ have been studied in many publications, especially in [7, 5, 8, 4]. We will now first give the equations for the surface $M_{ini}(0)$ which will serve as set of initial values at 0 and afterwards describe the results.

$$M_{ini}(0) := \left\{ (s, b_1, b_2) \in \mathbb{R}^3 \mid b_1^2 + \left( \frac{s^2}{4} - 1 \right) \cdot b_2^2 - 1 = 0 \right\}. \tag{4}$$

For a fixed $s \in ]-2, 2[$, the solution set of pairs $(b_1, b_2)$ for the describing equation consists of the two components of a hyperbola in the plane. For $s \in \{\pm 2\}$, it consists of two lines, for $s \in \mathbb{R} - [-2, 2]$ it is an ellipse. These curves glue together to a surface which is sketched in the left picture in Figure 1. It is roughly (= topologically) a sphere minus four holes. The right picture is given for later use. There the four holes are shifted to the front of the sphere, the back of the sphere is not visible.

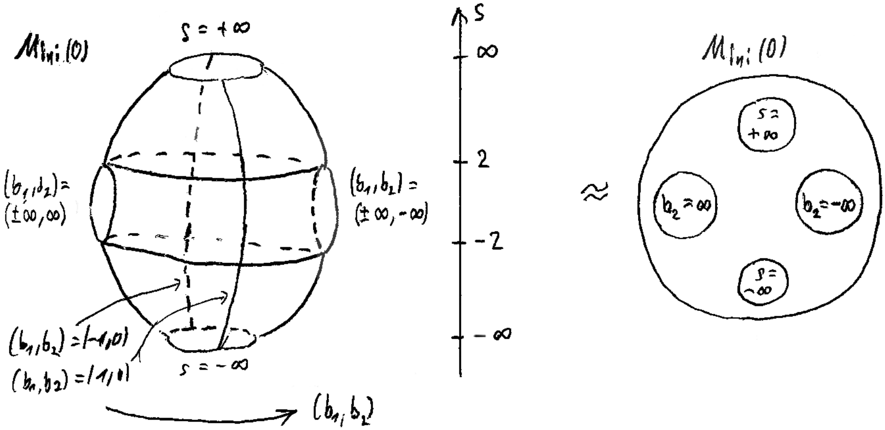


Figure 1: Two images of the surface $M_{ini}(0)$.

We claim that there is a natural bijection $M_{ini}(0) \to M_{sol}$, which we denote by $(s, b_1, b_2) \mapsto f_{(s,b_1,b_2)}$. To describe this bijection, we write down the *leading term*[1] of $f_{(s,b_1,b_2)}$ for $x$ near 0. It is in any case a simpler function, which approximates $f_{(s,b_1,b_2)}$ near 0 very well. In the following, we describe five different cases for the leading term, depending on the value of $s$.

1st case, $s \in ]-2, 2[$

The leading term of $f_{(s,b_1,b_2)}$ is

$$\frac{\Gamma(\frac{1}{2} - \alpha_-)}{\Gamma(\frac{1}{2} + \alpha_-)} \cdot b_- \cdot \left(\frac{x}{2}\right)^{\alpha_-}, \tag{5}$$

where $\alpha_- \in ]-\frac{1}{2}, \frac{1}{2}[$ and $b_- \in \mathbb{R}^*$ are defined by

$$\sin(\pi \alpha_-) = \frac{s}{2}, \tag{6}$$

$$b_- = b_1 + \sqrt{1 - \frac{s^2}{4}} \cdot b_2. \tag{7}$$

[1] Note that the notion of a leading term here is different from the more common notion of the leading term of a polynomial.

Note that in this case, we have $\frac{s}{2} \in ]-1,1[$ and $\sqrt{1-\frac{s^2}{4}} \in \mathbb{R}_{>0}$, so the equations (6) and (7) always have solutions. The expression $\Gamma(y)$ in equation (5) is the value at $y \in \mathbb{R}$ of the *Gamma function*, another classical transcendental function.[2] It can be calculated that near 0 the solution has no zeros or poles. The solution is either positive near 0 ($\iff b_- > 0 \iff b_1 > 1$) or negative near 0 ($\iff b_- < 0 \iff b_1 < -1$).

### 2nd case, $s = 2$

The leading term of $f_{(s,b_1,b_2)}$ is

$$2b_1 \cdot x \cdot \left(-\log\frac{x}{2} + \frac{\pi}{2} \cdot b_1 \cdot b_2 - \gamma_{Euler}\right), \tag{8}$$

here $\gamma_{Euler} = 0,5772\ldots$ is *Euler's constant*.[3] The solution has no zeros or poles near 0, it is either positive ($\iff b_1 = 1$) or negative ($\iff b_1 = -1$) there.

### 3rd case, $s > 2$

The leading term of $f_{(s,b_1,b_2)}$ is

$$-\frac{x}{t_{NI}} \cdot \sin\left(2t_{NI}\log\frac{x}{2} - 2\arg\Gamma(1+\sqrt{-1}\cdot t_{NI}) + \delta_{NI}\right), \tag{9}$$

here $t_{NI} \in \mathbb{R}_{>0}$ and $\delta_{NI} \in [0, 2\pi[$ are determined by

$$\cosh(\pi t_{NI}) = \frac{s}{2}, \tag{10}$$

$$\cos(\delta_{NI}) = b_1, \tag{11}$$

and $\Gamma$ denotes the Gamma function as before.

### 4th and 5th case, $s = -2$ and $s < -2$

The leading term of $f_{(s,b_1,b_2)}$ is the inverse of the leading term of $f_{(-s,b_1,-b_2)}$.

In all five cases, the formulas show that the initial value $(s, b_1, b_2) \in M_{ini}(0)$ determines the leading term. And it is not so hard to recover the initial value $(s, b_1, b_2)$ from the leading term and from the relation between $s$, $b_1$ and $b_2$ in equation (4). A hard fact is that the leading term determines the solution.

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[2] We do not need the precise definition of the Gamma function here. If you are interested, you can find it on Wikipedia: https://en.wikipedia.org/wiki/Gamma_function.

[3] If you are interested, you can find the definition of Euler's constant on Wikipedia: https://en.wikipedia.org/wiki/Euler%E2%80%93Mascheroni_constant.

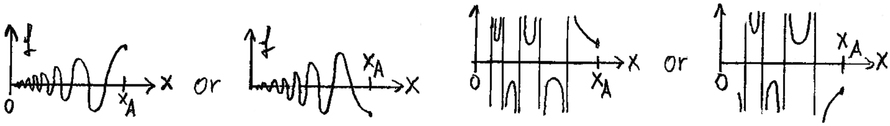
Therefore the initial values also determine the solution which gives a natural bijection $M_{ini}(0) \to M_{sol}$.

Roughly, the behavior of the asymptotics can be summarized as follows: There is a number $x_A = x_A(s, b_1, b_2) \in \mathbb{R}_{>0}$ such that $f_{(s,b_1,b_2)}(x_A) \neq 0$ and such that the following holds.

**1st, 2nd and 4th case:** The solutions with $s \in [-2, 2]$ have no zeros or poles in $]0, x_A]$, they are either positive or negative in the whole interval.

**3rd and 5th case:** The solutions with $s > 2$ do not have poles in $]0, x_A]$, but countably many zeros, alternatingly of types $[0+]$ and $[0-]$. See Figure 2a for a simple picture depicting this case.

The solutions with $s < -2$ do not have zeros in $]0, x_A]$, but countably many poles, alternatingly of types $[\infty+]$ and $[\infty-]$. See Figure 2b for a picture.




(a) $s > 2$ (b) $s < -2$

Figure 2: Sketches of the graphs of solutions for $|s| > 2$.

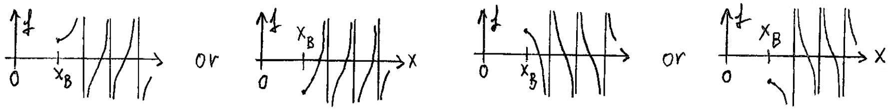
The important fact here is that the solution $f_{(s,b_1,b_2)}$ behaves in a determined way on the interval $]0, x_A]$. We can say that the interval $]0, x_A]$ is a *zone* on which the solution is controlled. Also note that this rough classification of the asymptotic behavior near 0 into three types uses only the parameter $s$, not the parameter $(b_1, b_2)$.

On the contrary, the types which turn up in the classification of the asymptotics near $\infty$ use the parameter $(b_1, b_2)$, but not the parameter $s$. Here we refrain from presenting precise formulas for leading terms and restrict the discussion to the rough behavior of the solutions near $\infty$: There is a number $x_B = x_B(s, b_1, b_2) \in \mathbb{R}_{>0}$ such that $f_{(s,b_1,b_2)}(x_B) \neq 0$ and such that the following holds.

**1st case**, $(b_1, b_2) = (\pm 1, 0)$: Then $f_{(s,b_1,b_2)}$ has no zeros or poles in $[x_B, \infty[$. In this interval the solution is positive for $b_1 = 1$ and negative for $b_1 = -1$.

**2nd case**, $(b_1, b_2) \neq (\pm 1, 0)$: If $b_2 > 0$ then $f_{(s,b_1,b_2)}$ has alternatingly zeros and poles of types $[0+]$ and $[\infty-]$. Figure 3a is a picture for this case.

If $b_2 < 0$ then $f_{(s,b_1,b_2)}$ has alternatingly zeros and poles of types $[0-]$ and $[\infty+]$. Figure 3b is a picture for this case.




(a) $b_2 > 0$ (b) $b_2 < 0$

Figure 3: Sketches of the graphs of solutions for $b_2 \neq 0$.

As before, the important fact is that the solution $f_{(s,b_1,b_2)}$ behaves in a determined way on the interval $[x_B, \infty[$. Our discussion so far allows us to define two zones (these are the intervals $]0, x_A]$ and $[x_B, \infty[$) such that we can qualitatively describe $f_{(s,b_1,b_2)}$ very well on each of the two zones. We will discuss the remaining interval $[x_A, x_B]$ in Section 4.

Table 1 summarizes the rough asymptotic behavior near 0 and $\infty$ of the solutions $f = f_{(s,b_1,b_2)}$ of equation (1) for all $(s, b_1, b_2) \in M_{ini}(0)$.

| | behavior on $]0, x_A]$ |
|---|---|
| $\lvert s\rvert \leq 2, b_1 \geq 1$ | $f(x) > 0$ |
| $\lvert s\rvert \leq 2, b_1 \leq -1$ | $f(x) < 0$ |
| $s > 2$ | $\dots[0+][0-]\dots$ |
| $s < -2$ | $\dots[\infty+][\infty-]\dots$ |

| | behavior on $[x_B, \infty[$ |
|---|---|
| $(b_1, b_2) = (1, 0)$ | $f(x) > 0$ |
| $(b_1, b_2) = (-1, 0)$ | $f(x) < 0$ |
| $b_2 > 0$ | $\dots[0+][\infty-]\dots$ |
| $b_2 < 0$ | $\dots[0-][\infty+]\dots$ |

Table 1: Qualitative behavior of solutions depending on initial parameters.

The type of the zero or pole which is closest to $x_A$ respectively $x_B$ depends on whether $f(x_A)$ respectively $f(x_B)$ is positive or negative.

## 4 Global geometry

The precise equations (5), (8), and (9) look hard, but Table 1 with the qualitative behavior is easy to grasp. The precise formulas are important in applications in mathematical physics, and their proofs use a lot of hard analysis. But Table 1 together with the qualitative behavior of the spaces $M_{ini}(x_0)$ allows the following stunning conclusion, and the proof of it uses just a good imagination and no calculations at all.

The conclusion is: *in Table 1 one can choose $x_A = x_B$!* That is, for every solution of equation (1), there is an $x_A \in \mathbb{R}_{>0}$ and an $x_B \in \mathbb{R}_{>0}$ such that we know the behavior of the solution in the zones $]0, x_A]$ and $[x_B, \infty[$. And now these $x_A$ and $x_B$ can be equal. This is surprising because a priori $x_A$ might be very small, and $x_B$ might be very large, and there might be a large *intermediate zone* where the sequence of zeros and/or poles is unknown and uncontrollable. The conclusion says that there is no intermediate zone.

Looking more carefully at Table 1, one finds: The combinations of the four subsets of $M_{ini}(0)$ in the first table and the four subsets of $M_{ini}(0)$ in the second table form altogether 14 sets (only the two combinations $|s| \leq 2, b_1 \geq 1$ and $(b_1, b_2) = (-1, 0)$, on the one hand, and $|s| \leq 2, b_1 \leq -1$ and $(b_1, b_2) = (1, 0)$, on the other hand, are impossible). So, there are 14 types of solutions. In one type the solutions are everywhere positive, in another they are everywhere negative, and the others are distinguished by the different global sequences of the four types of zeros and poles.

The stunning conclusion above follows from the Table 1 and the fact that the surfaces $M_{ini}(x_0)$ of initial values consist each of two open components of regular initial values and four lines of singular initial values. The part of the regular initial values in $M_{ini}(x_0)$ is isomorphic to $\mathbb{R}^* \times \mathbb{R}$, which consists of the two components $\mathbb{R}_{>0} \times \mathbb{R}$ and $\mathbb{R}_{<0} \times \mathbb{R}$. The four lines of singular initial values are the line of zeros of type $[0+]$, the line of zeros of type $[0-]$, the line of poles of type $[\infty+]$ and the line of poles of type $[\infty-]$. The details of the proof can be found in [4, Chapter 18]. But an idea can be obtained from the following pictures.

Figure 4 shows (like the right half of Figure 1) the front of the surface $M_{ini}(0)$, where the four holes are shifted to the front. There is a natural bijection $M_{ini}(x_0) \to M_{ini}(0)$ for any $x_0 \in \mathbb{R}_{>0}$. The images in $M_{ini}(0)$ of the four lines of singular initial values in $M_{ini}(x_0)$ for some $x_0 \in \mathbb{R}_{>0}$ are also shown in Figure 4.

Now we want to glue together all these surfaces $M_{ini}(x_0)$ for all possible $x_0 > 0$. Figure 5 shall give an idea of the resulting object $\bigcup_{x_0>0} M_{ini}(x_0) \cong \mathbb{R}_{>0} \times M_{ini}(0)$. The vertical lines $\mathbb{R} \times \{\text{point in } M_{ini}(x_0)\}$ represent the solutions $f_{(s,b_1,b_2)}$ of equation (1). In white regions they are positive, in gray regions they are negative. The four lines of singular initial values move, when $x_0$ moves, and Figure 5 tells how they move. Gluing all lines of one type gives four smooth surfaces which intersect the vertical lines transversally.

The three final pictures in Figure 6 shall give an idea of the solutions $f_{(s,b_1,b_2)}$ for some subfamilies of $M_{ini}(0)$ with fixed parameter $s$. We consider the three 1-parameter families with $s = s_0 > 2$, $s = s_1 \in ]-2, 2[$, and $s = s_2 = -s_0 < -2$. The families for $s = s_0$ and $s = s_2$ are ellipses, in Figure 6 these ellipses are cut at $(b_1, b_2) = (0, -\sqrt{s_{0/2}^2/4 - 1})$. The family for $s = s_1$ consists of two lines.

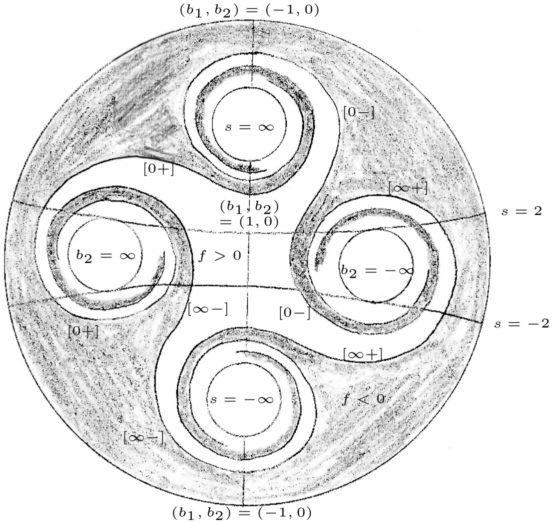
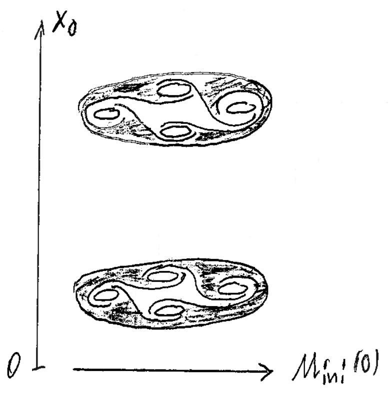



Figure 4: $M_{ini}(0) \cong M_{ini}(x_0)$ with the four lines in $M_{ini}(x_0)$.




$x_0$ large:
small spirals around $s = \pm\infty$
large spirals around $b_2 = \pm\infty$

$x_0$ small:
large spirals around $s = \pm\infty$
small spirals around $b_2 = \pm\infty$

Figure 5: How the four lines of singular initial values move when $x_0$ moves.

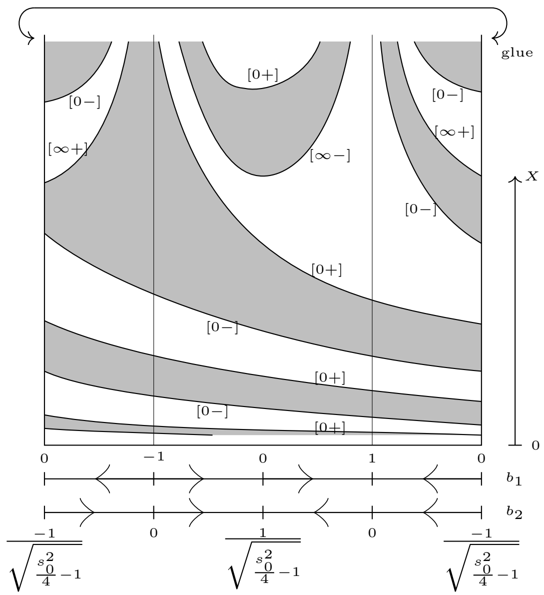
Let us have a closer look at how the three pictures in Figure 6 relate to each other. The symmetries in Figure 6 (the horizontal shift exchanges white and gray regions in the second picture, and the third picture is the mirror of the first) follow from the symmetries of the solutions,

$$f_{(s,b_1,b_2)} = -f_{(s,-b_1,-b_2)} = f^{-1}_{(-s,b_1,-b_2)}. \tag{12}$$

In the first and third picture, if $s_0$ or $s_2$ approach 2 or $-2$, the lines below the central regions (which contain open parts of the lines above $(1,0)$ and $(-1,0)$) approach the 0-line.

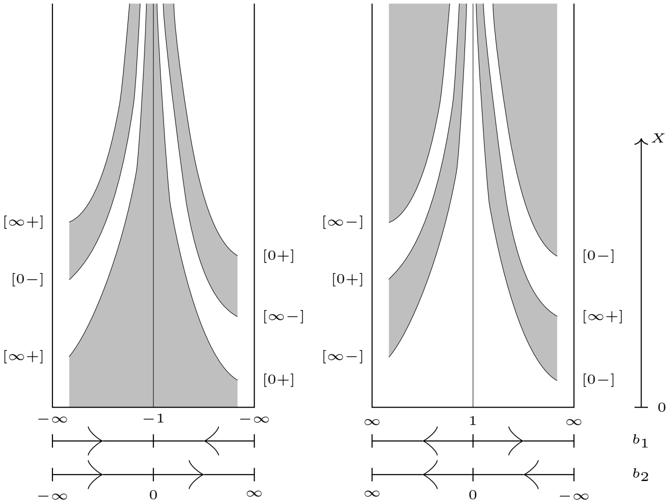
It is fun to try to see that the white regions in the three pictures glue together to a single white region in $\mathbb{R}_{>0} \times M_{ini}(0) \cong \bigcup_{x_0>0} M_{ini}(x_0)$. This exercise (together with Table 1) is also the key to the proof that the pictures in Figure 6 look as they look and that one can choose $x_A = x_B$ in Table 1.

For $x \to 0$ one stays in the first or third picture and has to wind around the hole $s = \infty$ or the hole $s = -\infty$ (for that, one has to go again and again from the left to the right respectively the right to the left). For $x \to \infty$ one has to wind around the hole $b_2 = \infty$ or the hole $b_2 = -\infty$. This requires going up and down through all three pictures.

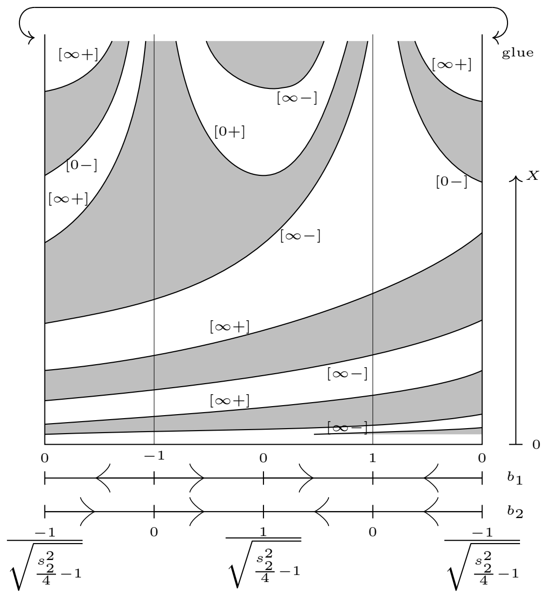



(a) $\mathbb{R}_{>0} \times M_{ini}(0)_{|s=s_0}$ for some $s_0 > 2$.

Figure 6: The families of solutions for $s \in \{s_0, s_1, s_2\}$.




(b) $\mathbb{R}_{>0} \times M_{ini}(0)_{|s=s_1}$ for some $s_1 \in ]-2, 2[$.




(c) $\mathbb{R}_{>0} \times M_{ini}(0)_{|s=s_2}$ for $s_2 = -s_0 < -2$.

Figure 6: The families of solutions for $s \in \{s_0, s_1, s_2\}$ (continued).

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