The codimension

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In this snapshot we discuss the notion of codimension, which is, in a sense, "dual" to the notion of dimension and is useful when studying the relative position of one object insider another one.

1 The notion of codimension

1.1 A simple observation

We start this snapshot by observing a crucial geometrical fact, which is at the origin of all our discussion. Fix a point $x \in \boldsymbol{R}$. Then pick two points $y_0, y_1 \in \boldsymbol{R}$ with y_0 on the left of x and y_1 on the right: there is no way we can join y_0 with y_1 with a smooth path in $\boldsymbol{R}$ without going through x. This fact would be no longer true if we had more space to move; for example if we regard the whole picture as placed in the plane $\boldsymbol{R}^2$. In this case, we could simply "go around" the obstacle x (see Figure 1).

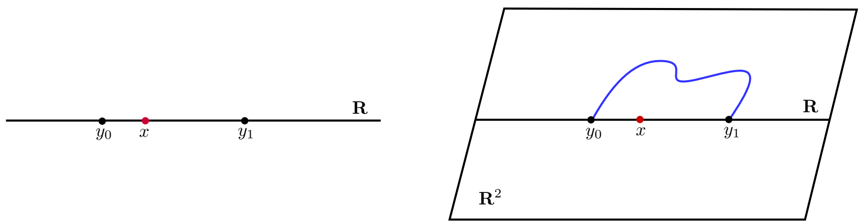

Figure 1: Joining two points y_0, y_1 on the real line, that are separated by another point x we wish to avoid, requires more "dimensions".

1.2 The difference between the dimensions

At the origin of this observation is a dimensional argument. A point is zero-dimensional, a line is one-dimensional and a plane is two-dimensional.[1] If X is a geometric object (for example a point or a circle) sitting inside another object Y (for example the line of real numbers, the plane $\boldsymbol{R}^2$ or 3-dimensional space), we call the difference between the dimension of Y and the dimension of X the *codimension* of X in Y, which formally reads

$$\operatorname{codim}_Y(X) = \dim(Y) - \dim(X).$$

We note that the codimension is a *relative* property: we always have to specify the ambient space in which we are working! For example, with this definition we get:

$$\operatorname{codim}_{\boldsymbol{R}}(\text{point}) = 1 \quad \text{and} \quad \operatorname{codim}_{\boldsymbol{R}^2}(\text{point}) = 2.$$

When an obstacle X has codimension at least *two* in the ambient space Y, we can reach any two points in $Y \backslash X$ without hitting X. Here by $Y \backslash X$ we mean the set of points in Y that do not belong also to X.

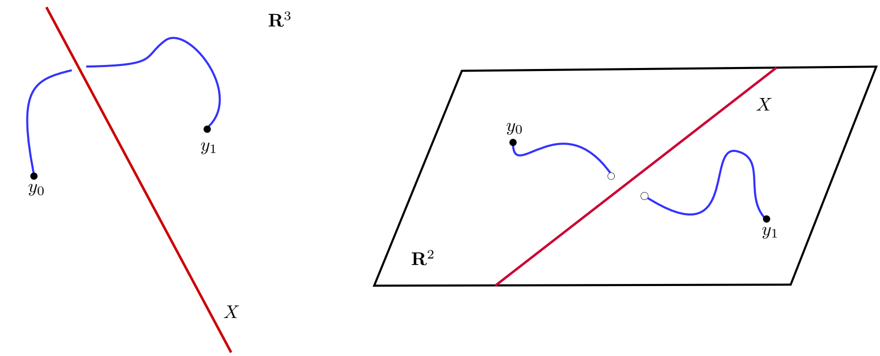

Figure 2: Left: A line (**red**) X has codimension *two* in the three-dimensional space $\boldsymbol{R}^3$. Two points $y_0, y_1 \in \boldsymbol{R}^3 \backslash X$ can always be joined (**blue** path passing under the straight one) avoiding X. Right: A line (**red**) X has codimension *one* in $\boldsymbol{R}^2$ and it might be impossible to join two points in $\boldsymbol{R}^2 \backslash X$ (broken **blue** path) without hitting X.

On the other hand, if $\operatorname{codim}_Y(X) = 1$, this might be false (see Figure 2). Moral of the story, with at least codimension 2 we can "go around obstacles".

[1] This is quite intuitive. For a more in-depth discussion about the concept of dimension see [2]

This is a consequence of a general result in the field of differential topology, which goes under the name of "Transversality theorem" (see the corollary on page 72 in [1]).

1.3 Links and knots

Let us look at the picture on the left of Figure 2. Because the infinite red line X (our "obstacle") has codimension two in $\boldsymbol{R}^3$ (our Y), we can join any two points in $\boldsymbol{R}^3 \setminus X$ without hitting X with the blue path. However, there are several different non-equivalent ways of joining these two points. For example, if we look at the picture on the left of Figure 3, we see that we cannot *continuously*[2] deform the blue path, joining y_0 and y_1, to the black path, which is wrapped around the line X without touching X. This has to do with the fact that the obstacle has codimension *two*. Similarly, consider a circle in $\boldsymbol{R}^3$ which is

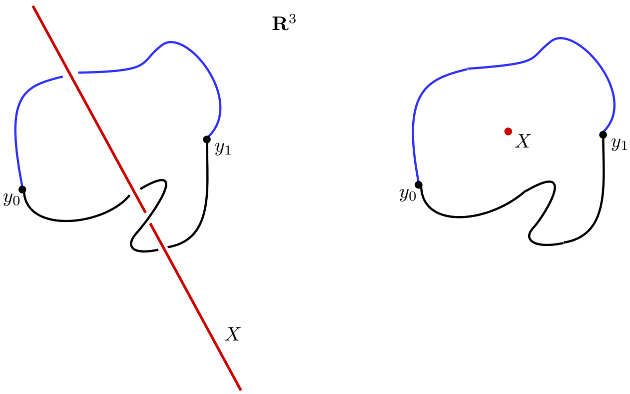

Figure 3: Left: two "non equivalent" ways (**blue** and black) of joining y_0 and y_1 without hitting X. It is impossible to continuously deform one of these two paths to match the other one without touching X. Right: if the (**red**) obstacle X was instead a point in $\boldsymbol{R}^3$, hence with codimension 3, this deformation would be possible.

"knotted" on the left picture in Figure 4. This is an example of a (nontrivial) "knot", called "trefoil". Is it possible to continuously deform it into a simple circle, like the one on the right of Figure 4? In general the answer is no. This occurs because the knot itself is a codimension two obstacle for such a

[2] By "continuously" here we mean that the line that joins y_0 and y_1 is held fixed at the two points and can be shrunk, lengthened and deformed like a very elastic string *without* detaching from the endpoints or cutting it.

deformation. However, it would be possible in $\boldsymbol{R}^4$, where the knot would have codimension three...

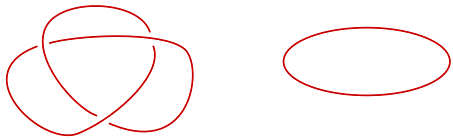

Figure 4: Left: The trefoil knot. Right: The trivial knot, also known as "unknot". We cannot deform the trefoil to the unknot using a process free from self-intersections.

1.4 The notion of general position

There is an idea closely related to the idea of codimension which turns out to be very powerful in geometrical arguments: the notion of general position (see [1]). We will illustrate this idea with a simple example – but we ask the reader all their attention!

Consider two bees flying in a dark room, ignoring each other. Their trajectories describe two curves in three-dimensional space. A very fundamental question that we can ask at this point is: *what is the probability that these curves intersect*? In other words, what is the probability that there is a point in space through which both bees pass during their flight (maybe at different times)?

Reasonably, this probability is zero, in the sense that if we do not impose special conditions on the trajectories (for example, if the bees were constrained to move on the same plane) then they will not intersect.

Similarly, we can ask what is the probability that the bees will hit a wall?[3] The reader might see that this probability is *not* zero. The intuition behind these arguments is based on the notion of codimension: the wall represents a codimension-one obstacle, which in general cannot be avoided. However, the trajectory of the other bee represents instead a codimension-two obstacle. This naive "probabilistic" idea is called in the mathematical language *general position*. A crude, but intuitive, definition of general position is that the trajectories are not "specially placed". Therefore, if the two trajectories are in general position, they will not intersect (see Figure 5).

Here are some properties we can derive based on a codimension reasoning:

* Two lines in general position in space do not intersect;

[3] Again, assuming that the bees fly ignoring the surrounding environment.

* Two curves in general position in the plane intersect only at some isolated points (the same is true for a curve and a surface in general position in three-dimensional space);
* A curve and a two-dimensional surface (for example a sheet of paper) in general position in four-dimensional space do not intersect.

Essentially: pick two objects "in general position" in n-dimensional space. If the sum of their codimensions is smaller than or equal to n, then it is likely that they intersect! If this sum is larger than n and they are in general position, they will not intersect.

An important consideration about general positions. *In the case of two ants moving on the plane, say a table, the probability that their trajectories intersect is* not *zero. At this point, however, we make an important subtle remark. Both for the bees and for the ants the probability that they pass at the same point* at the same time *is zero! This is because the time variable can be included in the problem as an additional dimension and therefore makes all the codimensions increase by one... Two curves in general position in three-dimensional space (the case of the ants in the plane with time) and in four-dimensional space (bees in space with time) do not intersect.*

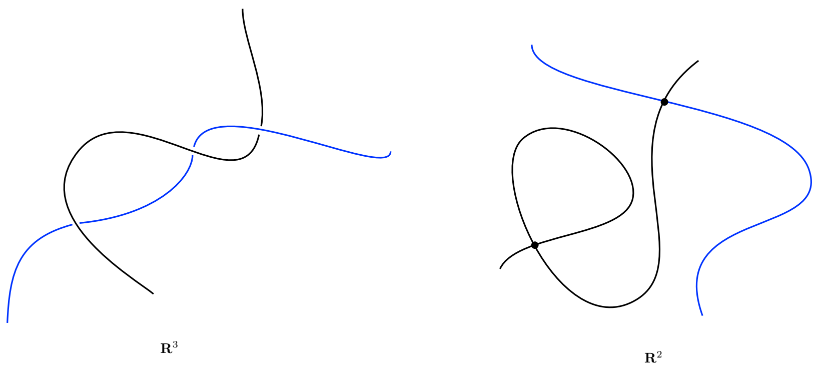

Figure 5: Left: Two curves in general position in $\boldsymbol{R}^3$ do not intersect. Right: Two curves in general position in $\boldsymbol{R}^2$ might intersect!

1.5 The codimension as "number of equations"

A useful way to think of the codimension of a geometric object is the number of equations needed to define it in the ambient space where it is placed.[4] For

[4] This intuitive discussion can be made rigorous introducing the notion of *regular* equation, see [1].

example: a line in the plane is defined by one equation, the well-known equation $y = a\,x + b$, where x, y are the coordinates on the plane and a, b are steepness and y-axis intersect of the line, respectively. Hence, it has codimension one. Similarly, a plane in three-space is defined by one equation. We need two equations to define a line in three-dimensional space: one first realizes that the line is the intersection of two planes and then takes the two equations defining these planes. One single equation in three-space (for example, the equation $x^2 + y^2 + z^2 = 1$ for a sphere) defines a surface, that is, a two-dimensional object with codimension one. An object defined by a single equation is called in general *hyper-surface*. The ambient space where a hyper-surface sits in can be possibly huge, but its codimension (the "relative dimension") is always one.

The reader must be aware, however, that for such considerations one equation over the complex numbers counts as two – one for the real and one for the imaginary part. Let's see an example. A complex line can be defined analogously to a real one: $y = a\,x + b$ where a, b and x are all complex numbers. Now it is possible to show that what one gets when considering a complex line is indeed a two-dimensional surface. Think about the plane of complex numbers: it consists of all numbers obtained by multiplying "1" by an element of $\mathbf{C}$, hence it is a complex line. A complex line has codimension two in $\mathbf{C}^2$ (it is a two-dimensional object in a four dimensional one), codimension four in $\mathbf{C}^3$ (because $\mathbf{C}^3$ can be thought as $\boldsymbol{R}^6$), and so on...

1.6 A related question

Here is a question to challenge the reader. Consider the two real polynomials $p_0(x) = x^2 - 1$ and $p_1(x) = x^2 + 1$. The zeores of p_0 are $1, -1$ and the zeroes of p_1 are $i, -i$, that is, $p_0(\pm 1) = p_1(\pm i) = 0$. Is it possible to continuously change the coefficients of p_0 into the coefficients of p_1, keeping them *real* and in such a way that, during this process, the zeroes remain *distinct*?

This question taps into the the concept of *homotopy* of polynomials. We say that, in a topological space, two functions are *connected by a homotopy* if there exists a continuous deformation process that transforms one into the other. A homotopy is, therefore, a continuous curve in the space of functions. The concept of homotopy can be better understood thinking about two curves. We say that two curves are homotopic if they can be continuously deformed one into another. We have already discussed such transformations above, regarding knots and trajectories of insects. Here we want to make this concept more abstract and discuss how it can help us, in this case, to answer questions such as the one about polynomials above.

We now look for a "family of polynomials" $p_t(x)$ of the general quadratic form $p_t(x) = a(t)x^2 + b(t)x + c(t)$, parametrized by the real parameter $t \in [0, 1]$ and with coefficients $a(t), b(t), c(t) \in \boldsymbol{R}$, which has the following properties:

- For $t = 0$ we have $a(0) = 1$, $b(0) = 0$ and $c(0) = -1$. That is, at $t = 0$ the polynomial is $p_0(x) = x^2 - 1$, the initial polynomial.
- For $t = 1$ we have $a(1) = 1$, $b(1) = 0$ and $c(1) = 1$. That is, at $t = 1$ the polynomial becomes $p_1(x) = x^2 + 1$, the final polynomial.
- The coefficients $a(t)$, $b(t)$ and $c(t)$ are continuous functions of t.
- Most importantly: the zeroes of p_t are distinct for all $t \in [0, 1]$. This means that if $x_1(t)$ and $x_2(t)$ are the two zeroes of p_t at any time t (namely, $p_t(x_1(t)) = p_t(x_2(t)) = 0$), then $x_1(t) \neq x_2(t)$.

Understanding the meaning of the question should be easier, now. The two zeroes x_t and x'_t are required to be distinct fo all t, and it is not obvious at all that such a family of polynomials p_t can exist.

Another interesting and related question is: what happens if, instead, we drop the condition that the coefficients $a(t)$, $b(t)$ and $c(t)$ must be real and we allow that, during the deformation, the coefficients can become complex? In the next section we will see how these questions are related to the notion of codimension. But before proceeding, try to guess the answer!

2 The codimension of multiple zeroes

2.1 Polynomials of degree two

Given the polynomial $p(x) = a\,x^2 + b\,x + c$, its *discriminant* is the number $\mathrm{disc}(p) = b^2 - 4ac$. From high-school, we know that the zeroes x_1, x_2 of p can be explicitly written in terms of the coefficients of p:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\mathrm{disc}(p)}}{2a},$$

and $x_1 \neq x_2$ (the zeroes are distinct) if and only if $\mathrm{disc}(p) \neq 0$.

Consider now the space of *all* degree-two real polynomials. This space can be naturally identified with the three-dimensional space $\boldsymbol{R}^3$ of coefficients:

$$\{\text{real polynomials } p(x) = ax^2 + bx + c\} \simeq \{(a, b, c) \mid a, b, c \in \boldsymbol{R}\} = \boldsymbol{R}^3.$$

We will use this identification for the remainder of this discussion: we think of a polynomial of degree two as a point in $\boldsymbol{R}^3$, whose coordinates are the polynomial's coefficients. Now, the discriminant $\mathrm{disc}(p)$ can be seen as a function on the space $\boldsymbol{R}^3$ of all real, degree-two polynomials. The set where this function takes the value *zero* is the hyper-surface $\Delta_{\boldsymbol{R}}$ of polynomials with two equal zeroes:

$$\Delta_{\boldsymbol{R}} = \{(a, b, c) \in \boldsymbol{R}^3 \mid b^2 - 4ac = 0\}.$$

This set is called the *discriminant variety* and is a two-dimensional infinite cone inside $\boldsymbol{R}^3$ (it is defined by a single equation, so its codimension is one, see Figure 6). Every polynomial that is not in $\Delta_{\boldsymbol{R}}$ (and therefore is in $\boldsymbol{R}^3 \backslash \Delta_{\boldsymbol{R}}$) has distinct zeroes; starting from a point that is not on the cone, as we approach $\Delta_{\boldsymbol{R}}$ the zeroes become closer and they coincide when we touch it.

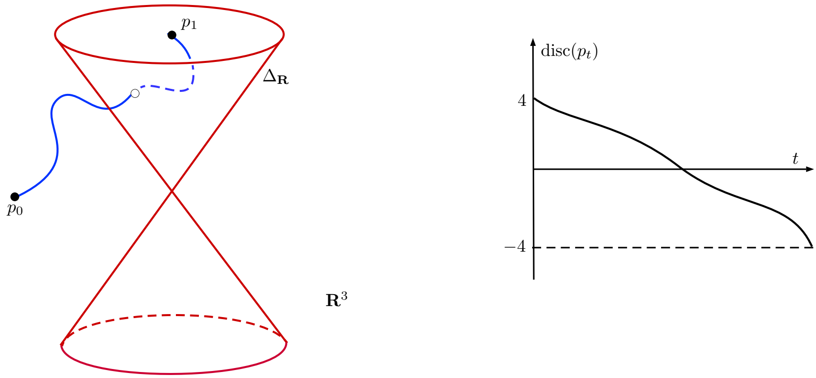

Figure 6: Left: the discriminant variety $\Delta_{\boldsymbol{R}} \subset \boldsymbol{R}^3$ and a path $t \mapsto p_t$ joining $p_0(x) = x^2 - 1$ with $p_1(x) = x^2 + 1$. Right: the function $t \mapsto \operatorname{disc}(p_t)$ must equal zero at some point $t \in (0, 1)$, since it is continuous.

Let us now go back to the question of Section 1.6, and start with a pictorial argument. The discriminant variety is an infinite cone. Therefore, if we pick a point p_0 outside the cone and a point p_1 inside it, we cannot go around the cone and we must cross it to connect p_0 to p_1 with a continuous curve (see Figure 6). Let us now look at a more mathematical description of this problem. Let the point p_0 correspond to the polynomial $p_0(x)$, while the point p_1 corresponds to the polynomial $p_1(x)$. If we try to join $p_0(x) = x^2 - 1$ to $p_1(x) = x^2 + 1$ with a continuous path p_t that represents the polynomials $p_t(x) = a(t)\,x^2 + b(t)\,x + c(t)$, with $t \in [0, 1]$, we must intersect the discriminant variety. The reason for this is the following: the continuous function $t \mapsto \operatorname{disc}(p_t)$ takes value 4 for $t = 0$ and -4 for $t = 1$. Consequently, it must take the value zero for some $t \in (0, 1)$. This is a consequence of the intermediate value theorem [3]. Therefore, we conclude that the answer to the first question of Section 1.6 is "no".

Let us move now to the same question over the complex numbers. The following family of *complex* polynomials joins p_0 with p_1 while keeping distinct roots (or zeroes):

$$p_t(x) = x^2 - e^{i\pi t} = (x + e^{\frac{i\pi t}{2}})(x - e^{\frac{i\pi t}{2}}).$$

Observe that, for this family of polynomials, we have $\mathrm{disc}(p_t) = 4e^{i\pi t} \neq 0$ and that the coefficients of the polynomial p_t are non-real for $t \neq 0, 1$. Intuitively p_t "rotates" around the *complex* discriminant variety:

$$\Delta_{\mathbf{C}} = \{(a, b, c) \in \mathbf{C}^3 \,|\, b^2 - 4ac = 0\}.$$

(Here we are identifying the space of all complex degree-two polynomials with $\mathbf{C}^3$ via their coefficients list.)

So, the question of Section 1.6 over the complex numbers has an affirmative answer. In fact, any two polynomials with *distinct* roots can be joined by a path in the space of complex polynomials *keeping distinct roots.* To see this, let us interpret the complex discriminant variety $\Delta_{\mathbf{C}}$ as an obstacle that we want to avoid. Remember what we said above about the number of equations and the codimension? The complex discriminant variety $\Delta_{\mathbf{C}}$ is defined by a single *complex* equation (equivalent to two real equations), hence it has codimension two in $\mathbf{C}^3$ and consequently we can always avoid it by going around it (see Figure 7)!

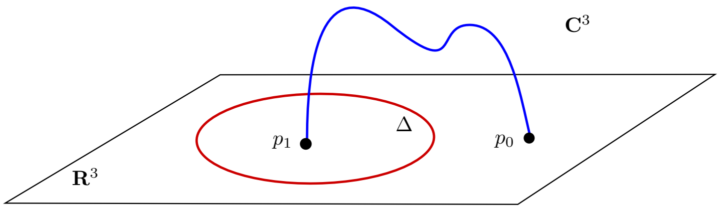

Figure 7: A simplified picture of the problem of avoiding the discriminant. In the space $\boldsymbol{R}^3$ the discriminant variety $\Delta_{\boldsymbol{R}}$ has codimension one, and it represents an obstacle (some sort of "wall") that might be impossible to avoid. Wen we move to the complex world the codimension of $\Delta_{\mathbf{C}}$ is two and we can simply turn around it.

2.2 The general picture

We can consider the following more general problem: given two real polynomials p_0 and p_1 of degree d with all distinct zeroes, is it possible to join them by a continuous path in the space of real polynomials keeping the zeroes distinct? Again the discriminant variety, that is, the hyper-surface of polynomials with multiple zeroes, acts as a wall separating the space of all real polynomials into many chambers. The answer to the previous question is “no” if the polynomials p_0 and p_1 lie in two different chambers and “yes” if the lie in the same chamber.

In the complex world the answer is always “yes”, because the complex discriminant has codimension *two* in the space of complex polynomials! Here we see the power of the notion of codimension: the space of all complex polynomials of degree d becomes very big when d increases (it has dimension $2d + 2$), and the same is true for the discriminant variety (which has dimension $2d$), but the only information we need to know is the difference between their dimensions.

References

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