Snapshots of modern mathematics
from Oberwolfach

№ 7/2018

# Tropical geometry

Erwan Brugallé • Ilia Itenberg
Kristin Shaw • Oleg Viro

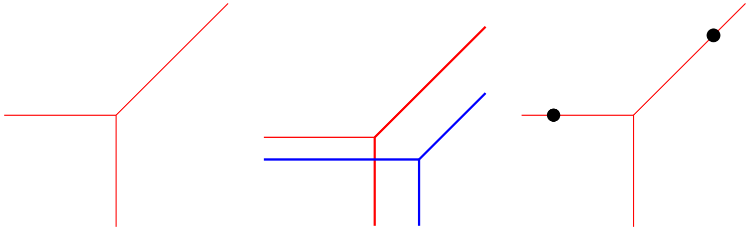
What kind of strange spaces hide behind the enigmatic name of tropical geometry? In the tropics, just as in other geometries, one of the simplest objects is a line. Therefore, we begin our exploration by considering tropical lines. Afterwards, we take a look at tropical arithmetic and algebra, and describe how to define tropical curves using tropical polynomials.

## 1 Entrée

This snapshot gives a glimpse of tropical algebra and geometry. To start with, let us take a look at one of the simplest objects from any geometry: a line.

A *tropical line* in the plane consists of three usual half lines in the directions $(-1,0),(0,-1)$ and $(1,1)$ emanating from some point in the plane (see Figure 1a). Why call this strange object a line, in the tropical sense or any other? If we look more attentively, we find that tropical lines have certain familiar geometric properties of "usual" or "classical" lines in the plane. For instance, most pairs of tropical lines intersect in a single point (see Figure 1b). Also for most choices of pairs of points in the plane there is a unique tropical line passing through the two points (see Figure 1c).

What is even more important, although not at all visible from the picture, is that classical and tropical lines are both given by an equation of the form $ax + by + c = 0$. In fact, one could say that classical and tropical geometries are developed following the same principles but from two different methods of calculation. They are simply the geometric faces of two different algebras.



a) b) c)

Figure 1: The tropical line

In the tropical world, addition is replaced by the operation of taking the maximum and multiplication is replaced by addition. Just by doing this, all of our objects drastically change form! In fact, even "being equal to 0" takes on a very different meaning.

Yet tropical geometry is not disconnected from the geometry we usually consider. In fact, many objects from classical geometry can be degenerated to tropical objects, and many tropical objects can be deformed, or "quantized", to classical ones. One advantage of this connection is that tropical objects are piecewise-linear and thus often they are much simpler to study than their classical counterparts! Additionally, if we keep track of the sometimes-peculiar behavior of geometric properties under these transitions we can study one of the two worlds with tools borrowed from the other. Even in a situation when such a deformation does not exist, this approach has been a rich source of conjectures.

Before diving into the subject, we should explain the use of the word "tropical". It is actually not due to the exotic forms of the objects under consideration. Before the term "tropical algebra", the more cut-and-dry name of "max-plus algebra" was used. Then, in honour of the work of their Brazilian colleague, Imre Simon, the computer science researchers at the *Université Paris Diderot* decided to trade the name of "max-plus" for "tropical".

# 2 Tropical algebra

## 2.1 Tropical operations

Tropical algebra begins with the set of real numbers, where addition is replaced by taking the maximum, and multiplication is replaced by the usual addition. In other words, we define on the set $\mathbb{R}$ of real numbers two new operations, called *tropical addition* and *tropical multiplication* and denoted by “ + ” and “ × ” respectively, in the following way:

$$\text{“}x + y\text{”} = \max(x, y), \qquad \text{“}x \times y\text{”} = x + y.$$

In this entire text, quotation marks are placed around an expression to indicate that the operations should be regarded as tropical. Just as in classical algebra we often abbreviate “$x \times y$” to “$xy$”. To familiarize ourselves with the use of these two operations, let us do some simple calculations:

$$\text{“}1 + 1\text{”} = 1, \quad \text{“}1 + 2\text{”} = 2, \quad \text{“}1 + 2 + 3\text{”} = 3, \quad \text{“}1 \times 2\text{”} = 3, \quad \text{“}1 \times (2 + (-1))\text{”} = 3,$$

$$\text{“}1 \times (-2)\text{”} = -1, \quad \text{“}(5 + 3)^2\text{”} = 10.$$

The tropical operations have many properties in common with the usual addition and multiplication. For example, they are both commutative:

$$\text{“}x + y\text{”} = \text{“}y + x\text{”}, \quad \text{“}x \times y\text{”} = \text{“}y \times x\text{”},$$

associative:

$$\text{“}x + (y + z) = (x + y) + z\text{”}, \quad \text{“}x \times (y \times z) = (x \times y) \times z\text{”},$$

and the tropical multiplication “ × ” is distributive with respect to the tropical addition “ + ”:

$$\text{“}(x + y) \times z\text{”} = \text{“}x \times z + y \times z\text{”}.$$

There is, however, no “tropical zero” in $\mathbb{R}$, that is to say tropical addition does not have an identity element in $\mathbb{R}$ (i.e., there is no element $x$ such that $\max\{y, x\} = y$ for all $y$ in $\mathbb{R}$). Nevertheless, we can naturally extend our two tropical operations to $-\infty$ by posing

$$\text{“}x + (-\infty)\text{”} = \max(x, -\infty) = x \quad \text{and} \quad \text{“}x \times (-\infty)\text{”} = x + (-\infty) = -\infty$$

for any $x$ in $\mathbb{R} \cup \{-\infty\}$. We use the notation $\mathbb{T}$ for $\mathbb{R} \cup \{-\infty\}$. These are the *tropical numbers*. We have seen that, after completing $\mathbb{R}$ by $-\infty$, the tropical addition has an identity element. On the other hand, a major difference remains between tropical and classical addition: no element of $\mathbb{R}$ has an additive “inverse”. Said in another way, tropical subtraction does not exist. Neither can we solve this problem by adding more elements to $\mathbb{T}$ to try to cook up additive inverses. In fact, “ + ” is *idempotent*, meaning that “$x + x$” $= x$ for all $x$ in $\mathbb{T}$. Our only choice is to get used to the lack of tropical additive inverses.

Despite this last point, the set $\mathbb{T}$ of tropical numbers equipped with the operations “ + ” and “ × ” satisfies all of the other properties of the classical addition and multiplication. For example, 0 is the identity element for tropical multiplication, and every element $x$ of $\mathbb{T}$ different from $-\infty$ has a multiplicative inverse “$\frac{1}{x}$” $= -x$. We say that $\mathbb{T}$ is the *tropical semi-field*.

## 2.2 Tropical polynomials

Having defined tropical addition and multiplication, we naturally come to consider functions of the form

$$P(x) = \text{“}\sum_{i=0}^{d} a_i x^i\text{”} \quad \text{with } a_i \in \mathbb{T}.$$

The above expression is a familiar polynomial, but it is interpreted tropically: the addition and multiplication are tropical and the coefficients are tropical numbers. If some coefficient $a_i$ of $P(x)$ is equal to $-\infty$, we can omit the corresponding monomial in the expression “$\sum_{i=0}^{d} a_i x^i$” (since $-\infty$ plays the role of tropical zero). In addition, since 0 is the identity element of tropical multiplication, we write “$x^i$” instead of “$0x^i$”.

By rewriting $P(x)$ in classical notation, we obtain $P(x) = \max_{i=0}^{d}(a_i + ix)$. Let us look at some examples of tropical polynomials:

$$\text{“}x\text{”} = x, \quad \text{“}1 + x\text{”} = \max(1, x), \quad \text{“}1 + x + 3x^2\text{”} = \max(1, x, 2x + 3),$$

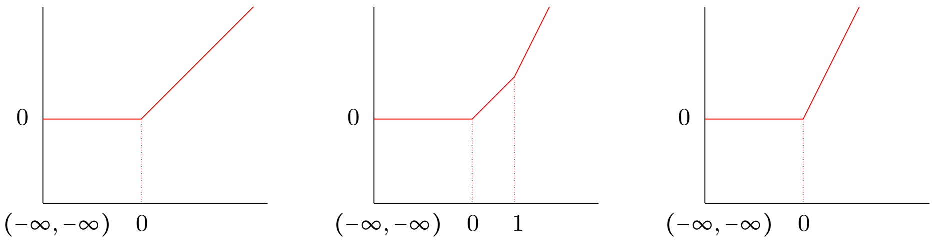
$$\text{“}1 + x + 3x^2 + (-2)x^3\text{”} = \max(1, x, 2x + 3, 3x - 2).$$

Each tropical monomial gives rise to an affine-linear function (a linear function plus a constant), and each tropical polynomial is the maximum of affine-linear functions.

Now let us find the roots of a tropical polynomial. Of course, we must first ask, what is a tropical root? The most standard definition of a classical root of a polynomial $P(x)$ is as follows: a root of $P(x)$ is a number $x_0$ such that $P(x_0) = 0$. If we attempt to replicate this definition in tropical algebra, we must look for elements $x_0$ in $\mathbb{T}$ such that $P(x_0) = -\infty$. However, if $a_0$ is the constant term of the polynomial $P(x)$ then $P(x) \geq a_0$ for all $x$ in $\mathbb{T}$. Therefore, if $a_0 \neq -\infty$, the polynomial $P(x)$ would not have any roots. This definition is surely not adequate.

We may take an alternative, yet equivalent, classical definition: a root of a polynomial $P(x)$ is a number for which there exists a polynomial $Q(x)$ such that $P(x) = (x - x_0)Q(x)$. We will soon see that a similar definition is appropriate for tropical algebra. To understand this, let us take a geometric point of view. A tropical polynomial is a piecewise-linear function (the graph of such a function is a broken line), and each piece has an integer slope (see Figure 2). What is also apparent from Figure 2 is that any tropical polynomial $P(x)$ is *convex*, that is, for any two points $(x_1, y_1)$ and $(x_2, y_2)$ of $\mathbb{R}^2$ such that $y_1 \geq P(x_1)$ and $y_2 \geq P(x_2)$, the segment connecting these two points is entirely contained in the *epigraph* $\{(x, y) \in \mathbb{R}^2 \mid y \geq P(x)\}$ of $P(x)$. This is because $P(x)$ is the maximum of a collection of affine-linear functions.

We call *tropical roots* of a tropical polynomial $P(x)$ all points $x_0$ of $\mathbb{T}$ for which the graph of $P(x)$ has a corner at $x_0$. Moreover, the difference in the slopes of the two pieces adjacent to a corner gives the *order* of the corresponding root. (The tropical number $-\infty$ is considered as a tropical root of $P(x)$ if the left-most segment of the graph of $P(x)$ is not horizontal; in such a situation, the slope of this segment is the order of the root $-\infty$.) Thus, the polynomial "$0 + x$" has a simple root at $x_0 = 0$, the polynomial "$0 + x + (-1)x^2$" has simple roots 0 and 1, and the polynomial "$0 + x^2$" has a double root at 0.




a) $P(x) =$ "$0 + x$"  b) $P(x) =$ "$0 + x + (-1)x^2$"  c) $P(x) =$ "$0 + x^2$"

Figure 2: Graphs of some tropical polynomials

The tropical roots, different from $-\infty$, of a tropical polynomial $P(x) =$ "$\sum_{i=0}^{d} a_i x^i$" $= \max_{i=0}^{d}(a_i + ix)$ are therefore exactly the tropical numbers $x_0$ for which there exists a pair $i \neq j$ such that $P(x_0) = a_i + ix_0 = a_j + jx_0$. We say that the maximum of $P(x)$ is attained (at least) twice at $x_0$. In this case, the order of the root at $x_0$ is the maximum of $|i - j|$ for all possible pairs $i$, $j$ which realize this maximum at $x_0$. For example, the maximum of $P(x) =$ "$0 + x + x^2$" is attained three times at $x_0 = 0$, and the order of this root is 2. Equivalently, $x_0$ is a tropical root of order at least $k$ of $P(x)$ if there exists a tropical polynomial $Q(x)$ such that $P(x) =$ "$(x + x_0)^k Q(x)$". Note that the factor $x - x_0$ in classical algebra gets transformed to the factor "$x + x_0$", since the tropical root of the polynomial "$x + x_0$" is $x_0$ and not $-x_0$.

As we have seen, two equivalent definitions of a classical root translate to two completely different notions in the tropical setting. In fact, this is a recurring problem in tropical mathematics: a classical notion may have many equivalent definitions, yet when we pass to the tropical world these could turn out to be different. In the case of roots of polynomials our second definition is much more interesting. In fact, with this definition, we have the following statement, which we encourage you to try to prove! The counterpart of this proposition when working over the complex numbers goes by the name of the *fundamental theorem of algebra*.

**Proposition 2.1** *The tropical semi-field $\mathbb{T}$ is algebraically closed. In other words, every tropical polynomial of degree $d \geq 1$ has exactly $d$ roots when counted with orders.*[1]

# 3 Tropical curves

## 3.1 Definition

Now let us increase the number of variables in our polynomials. A tropical polynomial in two variables can be written as $P(x,y)$ = "$\sum_{i,j} a_{i,j} x^i y^j$", or $P(x,y) = \max_{i,j}(a_{i,j} + ix + jy)$ in classical notation. In this way, our tropical polynomial is again a convex piecewise-linear function. Here for simplicity, we will only consider a tropical polynomial as a function on $\mathbb{R}^2$ and leave points with coordinates $-\infty$ aside. Then, the *tropical curve* $C \subset \mathbb{R}^2$ defined by $P(x,y)$ is the *corner locus* of this function. Said in another way, a tropical curve $C$ consists of all points $(x_0, y_0)$ in $\mathbb{R}^2$ for which the maximum of $P(x,y)$ is attained at least twice at $(x_0, y_0)$.

Let us look at the tropical line defined by the polynomial $P(x,y)$ = "$0+x+y$". We must find the points $(x_0,y_0)$ in $\mathbb{R}^2$ that satisfy one of the following three systems:

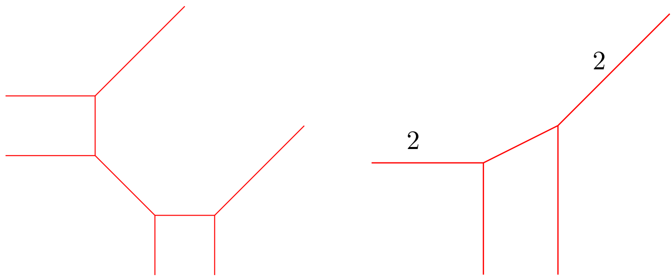
$$x_0 = 0 \geq y_0, \qquad y_0 = 0 \geq x_0, \qquad x_0 = y_0 \geq 0.$$

Hence, as depicted in Figure 1a, we see that our tropical line is made up of three standard half-lines

$$\{(0,y) \in \mathbb{R}^2 \mid y \leq 0\},\ \{(x,0) \in \mathbb{R}^2 \mid x \leq 0\},\ \text{and}\ \{(x,x) \in \mathbb{R}^2 \mid x \geq 0\}.$$

It is always the case that the corner locus of a tropical polynomial in two variables consists of line segments and half-lines, which we call *edges*. These intersect at points which we call *vertices*. However, we are still missing one bit of information to properly define a tropical curve. Just as in the case of polynomials in one variable (where we took the difference of the slopes to be the order of a root), for each edge of a tropical curve, we must take into account the difference in the slope of $P(x,y)$ on the two sides of the edge. If on the one side of an edge of the tropical curve, the function $P(x,y)$ coincides with the function defined by a single monomial "$a_{i,j}x^iy^j$", then the slope of $P(x,y)$ on that side is $(i,j)$. Therefore, we assign to each edge of a tropical curve the *weight* which is the greatest common divisor (gcd) of the numbers $|i-k|$ and $|j-l|$, where "$a_{i,j}x^iy^j$" and "$a_{k,l}x^ky^l$" are the monomials coinciding with $P(x,y)$ on two sides of the edge. Upon doing this, we have a complete definition of a tropical curve.

[1] The *degree* of a polynomial (in one variable) is the highest power occurring in it.




a) “$3 + 2x + 2y + 3xy + y^2 + x^2$” b) “$0 + x + y + y^2 + (-1)x^2$”

Figure 3: Two tropical conics

When depicting a tropical curve, the weight of an edge is only indicated if the weight is at least two. For example, in the case of a tropical line, all edges are of weight 1. Thus, Figure 1a represents fully a tropical line. Two examples of tropical curves of degree 2 are shown in Figures 3a and b. (The degree of a polynomial in several variables is the highest degree of its monomials, the degree of a monomial being the sum of the powers appearing in this monomial; tropical curves defined by polynomials of degree 2 are called *tropical conics*). The tropical conic in Figure 3b has two edges of weight 2.

### 3.2 Balanced graphs and tropical curves

Tropical curves in $\mathbb{R}^2$ have a very nice property, namely that a certain relation, known as the *balancing condition*, is satisfied at each vertex.

Let $\Gamma \subset \mathbb{R}^2$ be a graph whose edges are straight line segments and half-lines that have rational slopes and are equipped with positive integer weights. Let $v$ be a vertex of $\Gamma$ adjacent to the edges $e_1, \dots, e_k$ with respective weights $w_1, \dots, w_k$. Since every edge $e_i$ is contained in a line (in the usual sense) with rational slope, there exists a unique integer vector $\vec{v}_i = (\alpha, \beta)$ in the direction of $e_i$ such that $\gcd(\alpha, \beta) = 1$ (see Figure 4). We say that $\Gamma$ satisfies the *balancing condition* at the vertex $v$ if

$$\sum_{i=1}^{k} w_i \vec{v}_i = 0.$$

We say that $\Gamma$ is a *balanced graph* if it satisfies the balancing condition at each vertex.

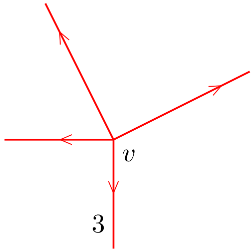



Figure 4: Balancing condition

In fact, it is possible to give a purely combinatorial characterization of tropical curves, without mentioning tropical polynomials.

**Theorem 3.1** *Any tropical curve in* $\mathbb{R}^2$ *is a balanced graph, and vice versa, any balanced graph in* $\mathbb{R}^2$ *represents a tropical curve.*

For example, the above theorem affirms that there exist tropical polynomials (in fact, of degree 3) whose tropical curves are the weighted graphs presented in Figure 5. In each case, the direction of the edges can be recovered from the direction of unbounded edges ($(-1,0)$, $(0,-1)$, and $(1,1)$) and the balancing condition.

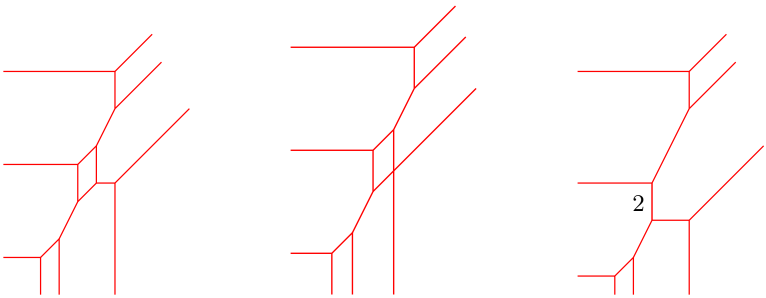



Figure 5: Three tropical cubics

## 4 To go further

Following our previous observations we could summarize one motivation to study tropical geometry as follows:

*Simple tropical objects provide useful information concerning their more complicated classical counterparts.*

As an example, let us mention generalizations of the two basic observations that we made in the introduction about tropical lines, namely that two lines intersect in one point, and that a single tropical line passes through two points. The first one generalizes to the *tropical Bézout Theorem* which states that, under some genericity assumption, two tropical curves of degree $d_1$ and $d_2$ intersect in $d_1 d_2$ points (counted with suitable multiplicities). The second one is the starting point of the development of *tropical enumerative geometry*, that had tremendous applications to classical (complex and real) enumerative geometry.

Other important applications of tropical geometry are in real algebraic geometry, combinatorics, mirror symmetry and mathematical biology, just to name a few.

We end this short introduction with some references that would allow an interested reader to delve deeper into the subject. The introductions to tropical geometry [BS14] and [SS09] do not require a serious background in mathematics. For a more advanced introduction to the subject, we refer to [IMS07], [Vir11], [BIMS15], and [MS15].

## References

[BIMS15] E. Brugallé, I. Itenberg, G. Mikhalkin and K. Shaw, *Brief introduction to tropical geometry*, in *Proceedings of the Gökova Geometry-Topology Conference 2014*, Gökova Geometry/Topology Conference (GGT), Gökova, 2015, pp. 1–75.

[BS14] E. Brugallé and K. Shaw, *A bit of tropical geometry*, American Mathematical Monthly **121** (2014), 563–589.

[IMS07] I. Itenberg, G. Mikhalkin and E. Shustin, *Tropical Algebraic Geometry*, Oberwolfach Seminars Series, Vol. 35, Birkhäuser, 2007.

[MS15] D. Maclagan and B. Sturmfels, *Introduction to Tropical Geometry*, Graduate Studies in Mathematics, Vol. 161, American Mathematical Society, 2015.

[SS09] D. Speyer and B. Sturmfels, *Tropical mathematics*, Mathematics Magazine **82** (2009), 163–173.

[Vir11] O. Ya. Viro, *On basic concepts of tropical geometry*, Proceedings of the Steklov Institute of Mathematics **273** (2011), 252–282.

Erwan Brugallé *is a professor at École Polytechnique in Paris.*
erwan.brugalle@math.cnrs.fr

Ilia Itenberg *is a professor at Université Pierre et Marie Curie - Paris 6 and École Normale Supérieure in Paris.*
ilia.itenberg@imj-prg.fr

Kristin Shaw *is an Alexander von Humboldt postdoctoral fellow at the Technische Universität Berlin.*
shaw@math.tu-berlin.de

Oleg Viro *is a professor at Stony Brook University, New York.*
oleg.viro@gmail.com

*Mathematical subjects*
Discrete Mathematics and Foundations, Geometry and Topology



*DOI*
10.14760/SNAP-2018-007-EN

*Snapshots of modern mathematics from Oberwolfach* provide exciting insights into current mathematical research. They are written by participants in the scientific program of the Mathematisches Forschungsinstitut Oberwolfach (MFO). The snapshot project is designed to promote the understanding and appreciation of modern mathematics and mathematical research in the interested public worldwide. All snapshots are published in cooperation with the IMAGINARY platform and can be found on www.imaginary.org/snapshots and on www.mfo.de/snapshots.

*Junior Editor*
Daniel Kronberg
junior-editors@mfo.de

*Senior Editor*
Carla Cederbaum
senior-editor@mfo.de

Mathematisches Forschungsinstitut
Oberwolfach gGmbH
Schwarzwaldstr. 9 – 11
77709 Oberwolfach
Germany

*Director*
Gerhard Huisken