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Topological recursion

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In this snapshot we present the concept of topological recursion – a new, surprisingly powerful formalism at the border of mathematics and physics, which has been actively developed within the last decade. After introducing necessary ingredients – expectation values, random matrices, quantum theories, recursion relations, and topology – we explain how they get combined together in one unifying picture.

1 Motivation and history

The formalism of topological recursion was discovered in [1, 4] and has been actively developed within the last decade. It already found a lot of applications in both mathematics and physics [1, 2, 3, 4, 5]. Indeed, the idea of topological recursion is quite interdisciplinary: While it originates from theoretical and mathematical physics, it also finds many applications in pure mathematics. Deep connections between mathematics and physics, which are uncovered by the topological recursion, provide one reason why this subject is so fascinating.

Roughly speaking, the topological recursion enables the computation of expectation values. Originally this recursion was developed in order to compute expectation values for particular (random) ensembles of matrices. However, very surprisingly, it turned out that in many seemingly unrelated fields, various expectation values can be computed following the same scheme, prescribed by the topological recursion. In other words, the topological recursion provides a universal formalism, or a set of tools, which can be taken advantage of independently of the origin or details of a given problem. This is the second reason why this formalism is so fascinating.

We start our presentation by explaining what kind of expectation values the topological recursion computes, and how it relates to probabilities in mathematics and to quantum theories in physics. Furthermore, as its name indicates, this formalism relates certain recursion relations with the field of mathematics known as topology. In the following we also explain what we mean by these notions. We conclude by stating how all these ingredients – expectation values, random matrices, quantum theories, recursion relations, and topology – get combined together in one unifying picture.

2 Expectation values and random matrices

In modern mathematics and physics, probability plays a fundamental role. In physics, this concept is particularly important in quantum theories. By the very nature of quantum physics, one can only indicate probabilities, or expectation values, of some events, rather than predict their precise outcome.

The *expectation value* can be seen as the long-run average of a quantity when considering many instances of an experiment to obtain this quantity. Let us quickly review how one computes the expectation value in the simplest example of rolling dice. If each face of a die (numbered by $k = 1, 2, \ldots, 6$) has the same probability $p(k) = \frac{1}{6}$ to end up on top when rolling the die then we can write the expectation value as

$$\langle \text{number on top face of die} \rangle := \sum_{k=1}^{6} k \cdot p(k) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \ldots + 6 \cdot \frac{1}{6} = 3\frac{1}{2}. \quad (1)$$

The symbol $\langle\ \rangle$ denotes the expectation value of some quantity, in our example of the number k on the top face when rolling dice. To compute the expectation value, we sum over the whole set of possible outcomes k, multiplied with their probabilities $p(k)$. The function p which assigns to each outcome k its probability $p(k)$ is also referred to as the *measure*. This probability measure is already normalized so that the probabilities of all outcomes get summed to 1. For a more general measure, this sum may not be 1. In this case, a normalization can be taken into account by dividing each expectation value by a constant Z which is the sum of the probabilities of all outcomes, or, equivalently, the expectation value of the constant quantity 1. In case of rolling dice, Z is automatically equal to 1:

$$Z = \langle 1 \rangle = \sum_{k=1}^{6} 1 \cdot p(k) = 6 \cdot \frac{1}{6} = 1. \quad (2)$$

Computations of expectation values in more involved examples, say in quantum theories, can be done analogously to the case of rolling dice – however, various ingredients of these computations are much more complicated. First of all, one typically deals with an infinite and continuous range of possible outcomes (instead of a finite number, such as six in rolling dice). Therefore the summation $\sum_{k=1}^{6}$ over the k outcomes has to be replaced by integration over some continuous variable M, which is denoted by $\int dM$. Furthermore, the probability $p(k)$ has to be replaced by an appropriate *integration measure*. In quantum theories, the integration measure takes the form of the function $e^{\frac{1}{\hbar}V(M)}$. This is the exponential of a certain function $V(M)$, called the *action*, divided by the parameter $\hbar$ referred to as the *Planck constant*[1]. With this more general notation, the overall normalization (2) takes the form

$$Z = \langle 1 \rangle = \int dM \, e^{\frac{1}{\hbar}V(M)}, \tag{3}$$

and it is referred to as the *partition function*. The partition function does not have to be identically equal to 1, it does not even have to be a constant. Instead, typically it is a function of some additional parameters, for example encoded in $V(M)$. Imagine that $V(M) = \sum_i t_i M^i$ is simply a polynomial in M – in this case these additional parameters could be identified with the coefficients t_i of this polynomial.

Rolling a die is a probabilistic system as the outcome can not be predicted with certainty. Another interesting class of probabilistic systems are random matrices. It turns out that random matrices provide interesting toy models of quantum systems and quantum field theory; a crucial property of quantum systems is that their evolution is indeterministic, and all we can compute are probabilities of various processes (for example where a particle is detected, how particles scatter, and so on). From a mathematical point of view, because of the probabilistic nature of quantum mechanics, a quantum theory can be regarded as some specific probability theory.

In those random matrix models, elementary degrees of freedom are represented by *matrices*, that is arrays of numbers of size $N \times N$, of the following form:

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[1] Here we call $\hbar$ the "Planck constant" because it appears in equation (3) in an analogous way as in the definition of the path integral in quantum mechanics. This name also indicates that $\hbar$ is a small number, appropriate to conduct a perturbative expansion. In physics, the Planck constant is one of the fundamental constants of Nature with an approximate value of $1.054 \cdot 10^{-34}\, J{\cdot}s$. It appears in the Heisenberg Uncertainty Relation (which states that simultaneous measurement of location and momentum of an object is impossible) and in many fundamental equations, for example the Schrödinger Equation.

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1N} \\ M_{21} & M_{22} & \cdots & \\ \vdots & \vdots & \ddots & \\ M_{N1} & M_{N2} & & M_{NN} \end{bmatrix}. \tag{4}$$

Matrices have analogous properties as ordinary numbers: they can be added, they can be multiplied, and so on. In particular, one can compute a polynomial $V(M) = \sum_i t_i M^i$ with a matrix argument M, or any other function, for example the exponential function. One also defines the *trace* of a matrix, denoted Tr, as the sum of all diagonal elements:

$$\mathrm{Tr}\, M = \sum_{k=1}^{N} M_{kk} = M_{11} + M_{22} + \ldots + M_{NN}. \tag{5}$$

It is particularly interesting for the field of quantum systems to consider matrices of very large size N. In this case, the inverse of the parameter N can be chosen very large and hence can for example be identified with the Planck constant, $\hbar = \frac{1}{N}$. Then the partition function (3) (generalizing (2)) takes the form

$$Z = \int dM\, e^{N\, \mathrm{Tr}\, V(M)}, \tag{6}$$

where $\int dM$ denotes appropriate integration over matrices and $V(M)$ is a polynomial (or some more complicated function) called the *potential.* One can also consider more involved expectation values – generalizing (1) in case of rolling dice – for example involving traces of powers of a matrix

$$\langle \mathrm{Tr}\, M^k \rangle = \int dM\, (\mathrm{Tr}\, M^k)\, e^{N\, \mathrm{Tr}\, V(M)}, \tag{7}$$

or products of such traces. Using the expression for a geometric series $\frac{1}{1-a} = \sum_{k=0}^{\infty} a^k = 1 + a + a^2 \ldots$, it follows that expectation values $\langle \mathrm{Tr}\, M^k \rangle$ can be identified as coefficients in a generating series

$$\Big\langle \mathrm{Tr} \frac{1}{x - M} \Big\rangle = x^{-1} \cdot \Big\langle \mathrm{Tr} \frac{1}{1 - Mx^{-1}} \Big\rangle = \sum_{k=0}^{\infty} x^{-k-1} \langle \mathrm{Tr}\, M^k \rangle. \tag{8}$$

This (x-dependent) expectation value is called the *resolvent.* More generally, one can consider *multi-resolvents*, defined as expectation values of products of traces of the above form. As such multi-resolvents essentially encode information about all possible expectation values, it is very useful to compute them in a random matrix model under consideration, or any other quantum or probabilistic theory which has features similar to matrix models.

However, it turns out that the computation of such multi-resolvents is very difficult. Instead of computing their exact form, one may try to compute their Taylor series expansion in the Planck constant $\hbar$,[2] or equivalently – in the context of matrix models – in powers of $\frac{1}{N}$. This is one reason why considering large sizes of matrices N is useful: for large N, its inverse $\frac{1}{N}$ is a small parameter, which can be considered as an expansion parameter. Taking N to be large[3] is called the *'t Hooft limit*, or simply *large N limit* of the family of matrix models. Computation of expectation values in such a limit is analogous to the computation of expectation values in the expansion in $\hbar$ in quantum mechanics, which is often referred to as the *WKB expansion* (named after Gregor Wentzel, Hendrik Anthony Kramers and Léon Brillouin).

Let us therefore consider a multi-resolvent that depends on several generating parameters $x_1, \ldots, x_n$ and consider its expansion[4] in powers of $\frac{1}{N}$:

$$\left\langle \left(\mathrm{Tr}\frac{1}{x_1 - M}\right) \cdots \left(\mathrm{Tr}\frac{1}{x_n - M}\right) \right\rangle = \sum_{g=0}^{\infty} N^{2-2g-n} \cdot W_n^g(x_1, \ldots, x_n). \tag{9}$$

It follows that to solve a random matrix model – or, more generally, some quantum or probabilistic theory – we need to compute all the coefficients $W_n^g(x_1, \ldots, x_n)$ in the above expansion. It turns out that the topological recursion provides a way to conduct such a computation, as we explain in what follows in Section 4. However, let us first recall some basic facts concerning recursion relations in general.

3 Recursion relations

We pause now the discussion of expectation values in random matrix models, and quickly recall what is meant by *recursion relations* in general. Such relations provide a recipe to compute successively a series of quantities such that each such quantity depends on those computed earlier. An example of quantities that can be computed recursively are Fibonacci numbers F_k: each of those numbers is the sum of the two preceding numbers, which can be written as

$$F_k = F_{k-1} + F_{k-2}. \tag{10}$$

[2] To compute a *Taylor series expansion in x (centered at 0)* means to express a function $f(x)$ in the form $\sum_{k=0}^{\infty} a_k x^k$. The geometric series that we used before is an example for a Taylor series expansion. When limiting this expression to the first terms, we get a polynomial which is an approximation of the function.

[3] Taking N to be large corresponds to taking the Planck constant $\hbar$ to be small.

[4] More precisely, the expression (9) is correct once a suitable notion of multi-point expectation value is introduced; this would require more technical discussion, which we skip in this note.

Introducing initial Fibonacci numbers as $F_0 = 0$ and $F_1 = 1$, from the relation (10) one can compute the other Fibonacci numbers one by one: $F_2 = 1 + 0 = 1$, $F_3 = 1 + 1 = 2$, $F_4 = 3$, $F_5 = 5$, and so on. Fibonacci numbers arise in many problems, not only in mathematics, but also in biology, where they describe for example the arrangement of leaves on a stem.

Another set of numbers, closely related to the main topic of these notes, are Catalan numbers, defined recursively by

$$C_k = \sum_{i=0}^{k-1} C_i \cdot C_{k-i-1} \tag{11}$$

and $C_0 = 1$. From this relation one can determine $C_1 = 1 \cdot 1 = 1$, $C_2 = 1 \cdot 1 + 1 \cdot 1 = 2$, $C_3 = 5$, $C_4 = 14$, and so on. Catalan numbers also arise in a plethora of problems, for example C_k is the number of ways in which a polygon with $k + 2$ sides can be divided into triangles.

4 Topological recursion for expectation values

We can finally explain what the topological recursion is. Recall that information about a random matrix model is essentially encoded in a set of expectation values $W_n^g(x_1, \ldots, x_n)$ defined in (9). It was shown in a couple of research papers such as [1, 4] that these expectation values can be computed recursively, using a relation that schematically takes the form

$$W_{n+1}^g \sim W_{n+2}^{g-1} + \sum_{i=0}^{g} \sum_{J \subseteq \{1,\ldots,n\}} W_{|J|+1}^{i} W_{n-|J|+1}^{g-i}, \tag{12}$$

where in the summations the terms $(i, J) = (0, \emptyset)$ and $(i, J) = (g, \{1, \ldots, n\})$ are omitted, and with appropriate initial conditions. This is the topological recursion that we have been after, and it enables to compute expectation values $W_{n+1}^g(x_1, \ldots, x_n)$ from the knowledge of expectation values whose upper index is not larger and whose lower index is at most one higher than the respective indices g and $n + 1$ of the expectation value W_{n+1}^g which we aim to compute. In Formula (12), the sum over J denotes the sum over all subsets of labels $1, 2, \ldots, n$.

To avoid various technical details, Formula (12) has been simplified in our presentation. In particular, the dependence of various terms W_n^g on $x_1, x_2, \ldots$ has been suppressed; to compute the precise dependence on x, in the right hand side of Formula (12) some additional residue computation must be performed. All such details are hidden behind the $\sim$ sign (which we wrote instead of $=$), and for our purposes it is not necessary to discuss them. The most important aspect we are interested in is how the indices g and n appear in the relation (12).

Note that if we set $n = 0$ then J is necessarily the empty set, so that the sum over J is irrelevant in Formula (12). If in addition we ignore the first term W^{g-1}_{n+2} in the right hand side in Formula (12) then this recursion reduces simply to the recursion for the Catalan numbers (11) (with shifted indices). It follows that the Catalan numbers provide an underlying structure of all models governed by the topological recursion (and in each such model there is a set of special expectation values that satisfy the Catalan recursion) – this is therefore one more important role that Catalan numbers play! For this reason, one can also regard the topological recursion as a highly non-trivial generalization of the Catalan recursion (11).

5 Why topological?

Finally we should explain what the recursion that we are discussing has to do with topology. Topology is a branch of mathematics that deals with features of objects such as surfaces and classifies them up to continuous deformations, that is scaling, squeezing, twisting but not cutting or gluing. For example, curved surfaces such as in Figure 1 are classified by how many "holes" or "handles" they have. The number of holes in a surface is called its *genus*. A sphere has genus zero, and topologically it is the same as, say, a cube or a tetrahedron. A donut has genus one, and topologically it is the same as a cup with a single handle. One aim of topologists' work is to classify geometric objects in various dimensions in a similar spirit, up to continuous deformations.

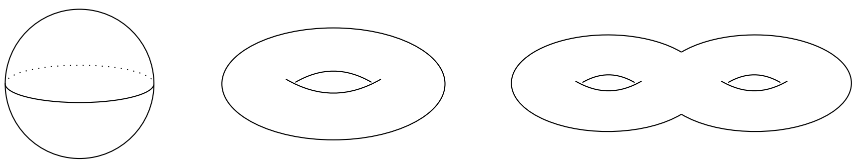

Figure 1: Surfaces of genus 0, 1, and 2.

It turns out that the coefficients W^g_n that appear in Formula (9) can be interpreted as characteristics of surfaces, with g denoting the genus of such a surface, and n the number of its punctures (marked points on the surface). With such an identification, the topological recursion (12) can be schematically represented as in Figure 2. The figure shows how a surface of a given genus and with a given number of punctures (represented by intervals) can be split up in different ways, by distributing the genus and punctures on two surfaces. Moreover, the first term in the right hand side in the formula for the topological recursion (12) represents an additional correction which is the first term on the right hand side of the equation in the figure.

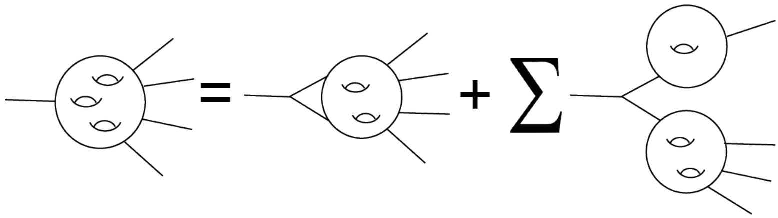

Figure 2: A schematic representation of the topological recursion. The sum sign in the last term represents the summation over all possible ways of splitting the genus and the number of punctures among two surfaces.

In fact this figure is not only a graphical analogy of the mathematical Formula (12) – it has deep meaning, especially in string theory. In string theory, one considers one-dimensional strings, which span two-dimensional surfaces (called “string worldsheets”) while moving in spacetime. Such strings can also interact, get connected or get separated. String theory is a quantum theory and to solve it one needs to compute probabilities of various such interactions. It turns out that in one particular class of string theory models, the graphical relation from Figure 2 represents interactions of strings moving in spacetime. In particular W_n^g denotes a probability of a certain process involving a string worldsheet of genus g. In those string theory models, the relation (12) enables recursive computation of probabilities of certain processes involving interactions of strings.

It is also important to note that the quantity W_n^g is defined in Formula (9) as a coefficient of N^{2-2g-n}, that is, for a fixed n it corresponds to a particular value of g, which appears as the power of N. This means that the genus g of surfaces represented in Figure 2 is determined by the power of N. In other words, the expansion of expectation values in powers of N has a topological interpretation. For this reason, the large N expansion is also often referred to as the *topological expansion*.

6 Summary

To sum up, we have presented the topological recursion as a set of recursion relations (12) that enable the computation of expectation values $W_n^g(x_1, \ldots, x_n)$ defined in Formula (9). Our presentation was based on a random matrix model interpretation, with N denoting the size of matrices. However, surprisingly, the same set of recursion relations (12) arises in many other problems in mathematics and physics, seemingly unrelated to random matrices. We already pointed out that the topological recursion has a beautiful interpretation in some particular class of string theory models, with the graphical representation in Figure 2 encoding string worldsheets. To mention just one purely mathematical application, it enables computation of various knot invariants, which characterize knots (such as those that can be tied with a piece of rope), and which are objects of great interest for knot theorists as can be seen for example in [2]. This and other applications of the topological recursion (for example, in quantization formalism or mirror symmetry) are discussed also in [5].

In the last few years, dozens of research papers have been written, and a number of conferences in various places – including Oberwolfach – have been organized, which attracted many mathematicians and physicists who are trying to understand the universal character and unexpected power of the topological recursion. The research in this topic is ongoing and we are convinced that many fascinating properties of the topological recursion are still to be uncovered. Everyone who is interested in interdisciplinary modern developments at the frontiers of mathematics and physics is encouraged to join our efforts!

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