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Triangulations in geometry: from Ptolemy to Teichmüller

Matthew Pressland

Ptolemy's theorem is a classical result from ancient Greek mathematics, concerning the lengths of sides and diagonals of a polygon drawn in a circle. In this snapshot, I will explain why this theorem is still important today through its role in Teichmüller theory, a subject which seeks to describe all possible "shapes" of a surface with boundary.

1 Ptolemy's theorem

It is a famous mathematical fact that any triangle is *circumscribed*, meaning it has a (unique) circle passing through its three corners. How easy it is to visualise this circle depends on the shape of the triangle—the centre of the circle may fall outside of the triangle, for example—but nevertheless it always exists. For shapes with four or more sides, this is no longer the case (see the rightmost image in Figure 1), and those which can be circumscribed have very special properties.

Ptolemy's theorem is a fact about circumscribed quadrilaterals. It gives the formula

$$xy = \alpha\gamma + \beta\delta$$

for the product of lengths of the two diagonals of such a quadrilateral in terms of the lengths of its sides, as shown in Figure 2. Ptolemy (100–170 CE) was an ancient Greek scientist, who discovered this theorem while studying astronomy.

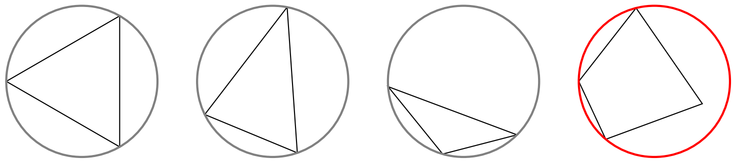

Figure 1: Every triangle, but not every quadrilateral, is circumscribed.

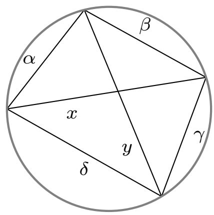

Figure 2: The lengths and diagonals of a circumscribed quadrilateral always satisfy $xy = \alpha\gamma + \beta\delta$; that is, the product of lengths of the diagonals is equal to the sum of products of lengths of opposite sides.

Rearranging the formula, we see that if we know x, α, β, γ, and δ (that is, the lengths of the sides and one of the diagonals of our circumscribed quadrilateral), we can compute the length y of the remaining diagonal via the formula

$$y = \frac{\alpha\gamma + \beta\delta}{x}.$$

Now consider a more general circumscribed polygon, with n sides. By choosing $n-3$ diagonals of this polygon which do not cross each other, we may divide it up into $n-2$ triangles, all inscribed in the same circle. This division is called a *triangulation*. For example, choosing either the diagonal labelled x or that labelled y in Figure 2 provides a triangulation of the quadrilateral. We cannot choose both x and y, since these diagonals cross: the four small triangles appearing in Figure 2 are not inscribed in the same circle as the original quadrilateral. A triangulation of a circumscribed pentagon is shown in Figure 3. The same figure shows how we may pass from one triangulation to another by "flipping" a chosen diagonal: any diagonal is shared by two triangles, and these two triangles together form a quadrilateral (whose sides may be either edges of the original polygon or other diagonals in the triangulation). Replacing the chosen diagonal by the other diagonal of this quadrilateral yields a new triangulation.

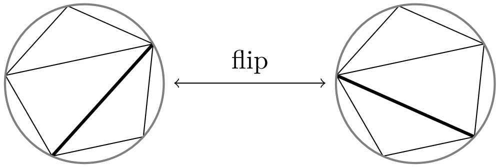

Figure 3: Flipping the bold diagonal in a triangulation of the pentagon.

If we choose a triangulation and measure the lengths of its diagonals (as well as those of the sides of the polygon), we can then start flipping, and use Ptolemy's theorem repeatedly to calculate the lengths of all of the other diagonals of the polygon.

2 Hyperbolic polygons

For polygons drawn on a flat piece of paper, Ptolemy's theorem may seem to be of limited interest. After all, most polygons are not circumscribed, and without this property, the theorem does not apply. However, the real power of Ptolemy's theorem appears when considering geometry on the *hyperbolic plane*, which is a curved surface. For an exploration of the foundations of hyperbolic geometry, see the snapshot [4]; here I will give a brief introduction.

On a flat surface, the shortest curve between two points is always a straight line. The hyperbolic plane is curved, so every path within it is as well, but any two points are still connected by a curve of shortest length, called a *geodesic*. Much as we draw distorted maps of our spherical planet on flat paper, we may represent the hyperbolic plane as a circle in which geodesics are given by circular arcs meeting the boundary at right angles (or, as a special case, straight lines passing through the centre of the circle); see Figure 4. This is known as the Poincaré disc model and was made famous by artist M. C. Escher (1898–1972) in his *Circle Limit* series.

Figure 4 (right) shows a tiling of the hyperbolic plane by triangles drawn in the Poincaré disc model. Despite appearances, the triangles do not actually get smaller near the boundary circle. This reflects the distortion inherent in compressing the infinite hyperbolic plane into a finite picture. We think of the bounding circle as being "at infinity." In particular, all of the arcs in Figure 4 have infinite length.

An *ideal polygon* in the hyperbolic plane consists of a set of points on the boundary circle, with neighbouring points connected by geodesics. An example of an ideal pentagon is shown in Figure 4 (centre). The word "ideal" here refers

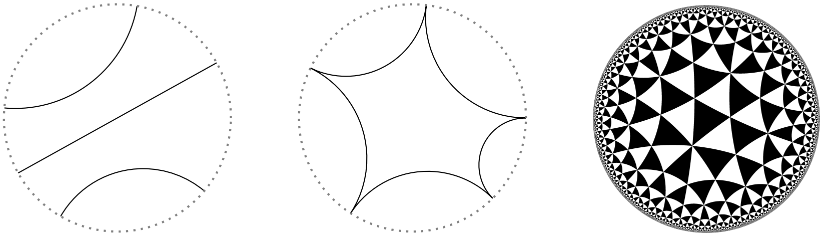

Figure 4: Geodesics in the Poincaré disc model of the hyperbolic plane (left), an ideal pentagon in this plane (centre), and a tiling of the hyperbolic plane related to Escher's *Circle Limit* series (right).

to the fact that the edges of the polygon do not actually meet in the hyperbolic plane, since the boundary circle is at infinity.

The ideal pentagon shown in Figure 4 is not very symmetric; its sides appear to have different lengths, even though they are all infinitely long in the hyperbolic plane itself. To make this intuition precise, we cut off the corners of the polygon using some additional curves called *horocycles*; see Figure 5. In the Poincaré disc, a horocycle is a circle tangent to the circle at infinity. Then for any side or diagonal of the ideal polygon, we may measure its length between the two horocycles. This allows us to compute a finite number, called the *lambda length* of the arc, after the Greek letter λ (lambda): if ℓ is the hyperbolic length of one of the bold curves in Figure 5, then the corresponding lambda length is e^{ℓ} if the horocycles do not intersect and $e^{-\ell}$ if they do.

Now something magical happens: these lambda lengths once again satisfy Ptolemy's theorem! Thus, just as for ordinary circumscribed polygons in the plane, if we fix a triangulation of the ideal polygon and a set of horocycles at its corners, the length of every diagonal in the polygon may be computed from the lengths of the arcs in the initial triangulation by flipping.

3 Teichmüller space

Our next question is the following: what are all the possible values of the lambda lengths of an n-sided ideal polygon in the hyperbolic plane? To answer this, we label the corners of our ideal n-gon by the numbers 1 to n (clockwise), and let λ_{ij} denote the lambda length of the geodesic from corner i to corner j. The set of all possible values of the λ_{ij}, from all possible ideal n-gons and choices of horocycles, is called the *decorated Teichmüller space* of ideal n-sided polygons. The adjective "decorated" refers to the fact that we choose the horocycles as

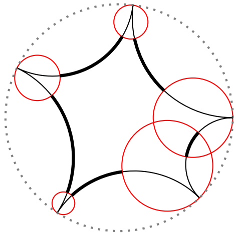

Figure 5: Horocycles at the corners of an ideal pentagon. These allow us to assign finite lengths to the sides or diagonals of the pentagon by measuring the signed distance along the curve between the two horocycles, shown here in bold.

well as just the polygon. Each "point" in this decorated Teichmüller space is a list of numbers which encodes information about a particular ideal n-gon and collection of horocycles. Of course, not just any list of numbers will do.

In general, a Teichmüller space is an example of a *moduli space*, a space whose points encode all the possible structures or configurations of some type of mathematical object. More specifically, a Teichmüller space is a moduli space of complex or hyperbolic structures, which can be thought of as coordinate systems. Often, moduli spaces have spaces have recognisable structures of their own. For instance, the Teichmüller space of nice coordinate systems on the torus can be seen as the upper half of the complex plane $\mathbb{C}$.

To work out what this particular decorated Teichmüller space is, we first observe that for any four distinct numbers i, j, k, and ℓ appearing in this order clockwise around the bounding circle, the polynomial equation

$$\lambda_{ik}\lambda_{j\ell} = \lambda_{ij}\lambda_{k\ell} + \lambda_{\ell i}\lambda_{jk}$$

must hold, because of Ptolemy's theorem. For example, for ideal quadrilaterals, the only such equation is $\lambda_{13}\lambda_{24} = \lambda_{12}\lambda_{34} + \lambda_{14}\lambda_{23}$.

However, not every solution to these equations gives a valid set of lambda lengths: for example, the values $\lambda_{13} = \lambda_{24} = -1$, $\lambda_{12} = \lambda_{34} = 1$, $\lambda_{14} = 0$, and $\lambda_{23} = -8$ satisfy the single equation when $n = 4$, but these cannot be the lambda lengths of an ideal quadrilateral since some of them are negative (or zero), and hence these values do not give us a point in Teichmüller space. So in addition to the equations coming from Ptolemy's theorem, we also require that the length $\lambda_{ij} \geq 0$ for all vertices i and j of the n-gon. Similarly, Ptolemy's equations have solutions involving complex numbers, but these also cannot represent lengths, and so we want to restrict to real number solutions.

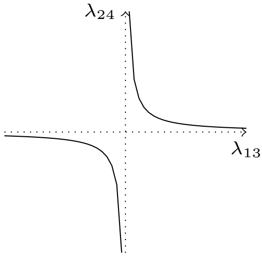

Figure 6: The Teichmüller space of ideal quadrilaterals with sides of length 1 is the set of positive real solutions to the equation $\lambda_{13}\lambda_{24} = 2$, that is, the upper-right piece of the plot.

These turn out to be all the conditions we need: that is, the Teichmüller space of ideal n-gons is exactly the set of all real and positive solutions to Ptolemy's equations. While we have explained above why these conditions are necessary, it is not obvious that any real and positive solution to Ptolemy's equations actually arises as the set of lambda lengths of an ideal n-gon, but this is nevertheless true.

In particular, the decorated Teichmüller space of ideal quadrilaterals consists exactly of real and positive solutions to the equation $\lambda_{13}\lambda_{24} = \lambda_{12}\lambda_{34} + \lambda_{14}\lambda_{23}$. We cannot easily draw this set, since it lives in a six-dimensional space, but Figure 6 shows the set of real solutions for which $\lambda_{12} = \lambda_{23} = \lambda_{34} = \lambda_{41} = 1$. This consists of two disconnected pieces, but only in one of them are the values λ_{13} and λ_{24} positive, and so this piece is the decorated Teichmüller space of ideal quadrilaterals with sides of length 1.

4 Computing with flips

If n is larger than 4, we are presented with a system of many equations. To help us find real and positive solutions practically, it is useful to change perspective slightly. As we discussed in Section 2, once we know the lambda lengths of the arcs in a triangulation of the n-gon, all of the others can be computed by flipping diagonals and applying one of Ptolemy's equations.

With this in mind, we pick a triangulation of the n-gon, which we call $\mathbb{T}$; a simple example is shown in Figure 7. Any triangulation consists of $2n-3$ arcs (including the sides of the n-gon), and so we get a list of $2n-3$ positive real numbers, that is, a point in $\mathbb{R}^{2n-3}_{>0}$, by taking the lambda lengths of just the arcs in $\mathbb{T}$ and forgetting the rest.

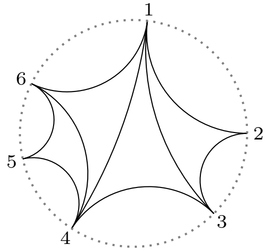

Figure 7: A triangulation of an ideal hexagon.

Now we try to reverse this procedure: given a list of $2n-3$ positive real numbers, can we find an ideal n-gon such that the arcs in $\mathbb{T}$ have precisely these numbers as lambda lengths?

If $\mathbb{T}$ contains the arc from corner i to j, let's write ℓ_{ij} for the lambda length we wish to give it. All together, we have $2n-3$ of these initial lengths ℓ_{ij}. We can use these to compute the remaining lambda lengths, one at a time. Since none of the arcs in $\mathbb{T}$ cross each other, none of Ptolemy's equations involve only the lambda lengths of these arcs, and so these equations do not put any constraints on our choice of initial lengths ℓ_{ij}. Instead, as in Section 2, by repeated flipping we can deduce what the lambda lengths of all the other arcs in the n-gon must be in terms of the desired lengths ℓ_{ij} for the arcs in $\mathbb{T}$. As discussed in Section 3, we just have to check that Ptolemy's equations do not force some of these other values to be zero, negative, or even complex.

However, this turns out not to be an issue. When applying Ptolemy's equation ($y = \frac{\alpha\gamma+\beta\delta}{x}$) to compute a new length after a flip, we already know the values on the right-hand side, and want to compute the remaining value y. But the expression only involves multiplication, addition, and division, and so y *must* be real and positive when the other values are. For example, starting from the triangulation in Figure 7, we compute that

$$\lambda_{24} = \frac{\lambda_{12}\lambda_{34} + \lambda_{23}\lambda_{14}}{\lambda_{13}} = \frac{\ell_{12}\ell_{34} + \ell_{23}\ell_{14}}{\ell_{13}},$$

and so λ_{24} is again real and positive, because all of our chosen values ℓ_{ij} are. Continuing in this way, we see that our computations will never leave the set of positive real numbers. It is thus at least plausible that there is an ideal n-gon with the correct lambda lengths, and with a bit more work, one can prove that such an n-gon really exists. We may thus summarise this result as follows.

Theorem. *Each triangulation $\mathbb{T}$ of an n-gon determines a one-to-one correspondence between points in the decorated Teichmüller space of ideal n-gons and points in the set $\mathbb{R}_{>0}^{2n-3}$ of lists of $2n-3$ positive real numbers.*

To illustrate the theorem in the quadrilateral case, we again look at Figure 6, in which the upper right part of the plot is the Teichmüller space of ideal quadrilaterals with sides of length 1. Since $n = 4$, we want to show that the decorated Teichmüller space corresponds to $\mathbb{R}_{>0}^{5}$, and if we fix the four lambda lengths of the sides of the quadrilateral to 1, that leaves only one degree of freedom. For our triangulation $\mathbb{T}$, we have to decide between using the diagonal from 1 to 3 or that from 2 to 4. In the first case, we associate a point in the Teichmüller space to its λ_{13} coordinate, which is a positive real number. In the second, we instead associate the point to its λ_{24} coordinate. In either case, we produce a one-to-one correspondence between the Teichmüller space and the positive real line $\mathbb{R}_{>0}$, as the theorem predicts.

5 Further reading

The decorated Teichmüller space (not just of a disc, but also of a more complicated surface with boundary) was defined by Robert Penner [7, 8], who proved in general that this space is in one-to-one correspondence with $\mathbb{R}_{>0}^{N}$, where N is the number of arcs needed to triangulate the surface. Ptolemy's equations are examples of exchange relations in Fomin and Zelevinsky's cluster algebras [3], a much bigger family of equations with positivity properties like those we used in Section 4. The connection between cluster algebras and Teichmüller space has been developed further by Gekhtman, Shapiro, and Vainshtein in [5], Fock and Goncharov in [1], and Fomin, Shapiro, and Thurston in [2].

Solutions to Ptolemy's equations over the positive *integers* (whole numbers), or equivalently decorated ideal n-gons with integer lambda lengths, are very closely related to frieze patterns, the subject of an earlier snapshot [6]. Indeed, a frieze pattern with $n-1$ rows, as in [6, Fig. 3], is one way of representing an integer point in the Teichmüller space of ideal n-gons.

Image credits

Figure 4 (right) "H2 Checkers 334". Author: Antonissimo. Entered into the public domain via Wikimedia Commons, https://commons.wikimedia.org/wiki/File:H2checkers_334.png, visited on January 12, 2026.

References

[1] V. V. Fock and A. B. Goncharov, *Cluster ensembles, quantization and the dilogarithm*, Annales Scientifiques de l'École Normale Supérieure. Quatrième Série **42** (2009), no. 6, 865–930, http://dx.doi.org/10.1007/978-0-8176-4745-2_15.

[2] S. Fomin, M. Shapiro, and D. Thurston, *Cluster algebras and triangulated surfaces. I. Cluster complexes*, Acta Mathematica **201** (2008), no. 1, 83–146, https://doi.org/10.1007/s11511-008-0030-7.

[3] S. Fomin and A. Zelevinsky, *Cluster algebras I: Foundations*, Journal of the American Mathematical Society **15** (2002), no. 2, 497–529, http://dx.doi.org/10.1090/S0894-0347-01-00385-X.

[4] H. Gangl, *Jewellery from tessellations of hyperbolic space*, Snapshots of modern mathematics from Oberwolfach (2022), no. 05, https://doi.org/10.14760/SNAP-2022-005-EN.

[5] M. Gekhtman, M. Shapiro, and A. Vainshtein, *Cluster algebras and Weil–Petersson forms*, Duke Mathematical Journal **127** (2005), no. 2, 291–311, https://doi.org/10.1215/S0012-7094-04-12723-X.

[6] T. Holm, *Friezes and tilings*, Snapshots of modern mathematics from Oberwolfach (2015), no. 04, https://doi.org/10.14760/SNAP-2015-004-EN.

[7] R. C. Penner, *Decorated Teichmüller theory of bordered surfaces*, Communications in Analysis and Geometry **12** (2004), no. 4, 793–820, http://projecteuclid.org/euclid.cag/1098468019.

[8] ———, *Decorated Teichmüller theory*, QGM Master Class Series, European Mathematical Society (EMS), Zürich, 2012, https://doi.org/10.4171/075.

Matthew Pressland *is a Maître de Conférences at the Université de Caen Normandie.*

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Mathematisches Forschungsinstitut
Oberwolfach gGmbH
Schwarzwaldstr. 9 – 11
77709 Oberwolfach
Germany

Director
Gerhard Huisken