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Alternating sign matrix bijections: marvelous, mysterious, missing

Jessica Striker[1]

A bijection transforms one type of mathematical object into another. Such transformations provide new perspectives on these objects, revealing surprising properties and uncovering new mysteries. We discuss bijections from alternating sign matrices to other objects in mathematics and physics and recent progress in the search for a missing bijection.

1 Bijections in real life and combinatorics

Suppose I hand you a glass of water and some ice cubes and ask, "Are these the same thing?" You might say, "No, one is solid and the other is liquid." Or you might say, "Yes, they are both composed of H_2O molecules." Which reply is correct? The answer is: both are! Although ice and liquid water look different, they are composed of the same substance, and there is a way to transform one into the other *and* get back again. Ice transforms to liquid water by sitting at room temperature, and liquid water transforms back to ice by being put in the freezer. Mathematically speaking, this is a *bijection*: a reversible transformation from one set to another. In *combinatorics*,[2] we are concerned with bijections between *finite* sets. Such a bijection can only exist if the two sets have the

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[2] Combinatorics is often called the art and science of counting.

$$
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

The permutation matrix corresponding to 46315827.

The same permutation matrix as a rook placement (where pawns were used instead of rooks).

Figure 1

same number of elements. Many combinatorial bijections give us new insights on hidden features of the objects.

Let us look at an example. A *permutation of n* is a rearrangement of the numbers $1, 2, \ldots, n$. For example, the six permutations of 3 are 123, 132, 213, 231, 312, and 321. There is a bijection between permutations of n and n-by-n square arrays of numbers (*matrices*) with one 1 in each row and column and all other entries equal to 0. Figure 1 (left) shows the matrix form of the permutation 46315827. Notice how the 1 in the first row of the matrix is in column 4, the 1 in the second row is in column 6, etc. Both the one-line notation 46315827 and the matrix are useful ways of looking at the same permutation. The first one helps you know how many ways there are for a group of eight children to form a line. The second helps you model placing the maximum number of mutually antagonistic rooks on a chessboard so that none attack each other; see Figure 1 (right).[3]

2 Alternating sign matrices

Now imagine we have black and white rooks on a chessboard and we want to place them so that no rook is attacking another rook of the same color. So if there were more than one rook in a row, the rooks would need to alternate in color. Suppose also that there is one more black than white rook in each row and in each column. This would be a weird way to play chess, but it makes a fun mathematical object called an alternating sign matrix!

An *alternating sign matrix (ASM)* is a square matrix whose entries are equal

[3] A rook attacks another rook if it is in the same row or the same column and there is no other chess piece in between.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

An alternating sign matrix.

The same alternating sign matrix as a two-color rook placement.

Figure 2

to 0, 1, or -1 and such that the sum of the entries in each row and in each column equals 1. Also, the non-zero entries need to alternate in sign along each row and each column. You can see a bijection to the weird rook placements by saying that each 1 is a black rook, each -1 is a white rook, and each 0 is a spot with no chess piece. See Figure 2, where the 1's in the matrix are represented by eight black pawns and two black rooks. Notice that the set of alternating sign matrices contains the set of permutation matrices, since a permutation matrix corresponds to a placement with only black rooks. With the above definition in mind, try to figure out which two of the following four matrices are an ASM! (The answer is at the end.)

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

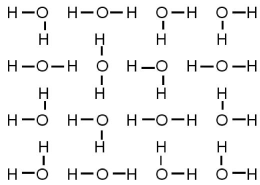
Alternating sign matrices have a fascinating history (see [3] for more details). They were discovered in 1984 by mathematicians who were studying a generalization of the determinant from linear algebra. They asked a very reasonable question: How many n-by-n alternating sign matrices are there? After doing some computations by hand and on the computer, they found that their data formed a pattern and *conjectured* (guessed) [8] that the number of n-by-n alternating sign matrices equals

$$\frac{1!\,4!\,7!\,\cdots(3n-2)!}{n!\,(n+1)!\,\cdots(2n-1)!},$$

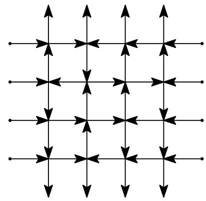
where $n! = n(n-1)\cdots 3\cdot 2\cdot 1$. For example, there are $\frac{1!\,4!\,7!}{3!\;4!\;5!} = 7$ different 3-by-3 ASMs. But it took 13 years to prove this counting formula! One of the proofs [7] relied on a beautiful bijection to square ice configurations, which we describe next.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

An ASM.



The corresponding square ice configuration.



Its six-vertex configuration.

Figure 3

3 Known alternating sign matrix bijections

3.1 Square ice

Back to ice. Imagine we had a grid full of hydrogen and oxygen atoms, where the hydrogen atoms lie in between the oxygen atoms, and with hydrogen atoms on the left and right sides. We'd like to draw bonds between the atoms to make them all into water molecules. We don't allow the atoms to move, so each oxygen needs to be connected to exactly two of the hydrogen atoms directly next to it. This forms a sheet of *square ice*; see Figure 3 (center).[4] You can get from a square ice configuration to an alternating sign matrix by the following recipe: each horizontal molecule corresponds to a 1, each vertical molecule to a -1, and all other molecules to a 0. To get from an ASM to a square ice configuration is slightly trickier, since there are multiple molecule configurations corresponding to a 0. But it turns out that if you draw the vertical and horizontal molecules first, at each other spot, there's a right choice of the four possible right angle molecules so that no hydrogens end up alone (try this out using the example in Figure 3!).

Usually, physicists study a graph in bijection with a square ice grid, where you replace each O by a dot ("vertex") and each H by an arrow ("directed edge") pointing to its connected O. In addition, you have to add upward-pointing arrows on the top and downward-pointing arrows on the bottom; see Figure 3 (right). This is called the *six-vertex model*, because there are exactly six possibilities for how the graph looks at a vertex in this situation; see Figure 4 (top and center). This beautiful bijection proved useful for enumerating ASMs [7], since it provided access to tools from physics.

[4] The reader well-versed in chemistry might note that frozen water molecules form a 120° angle rather than 90° or 180°. So these square ice configurations are not an accurate model of frozen ice. But, configurations such as these were found in graphene, so this is a physics model that shows up in the real world. [1]

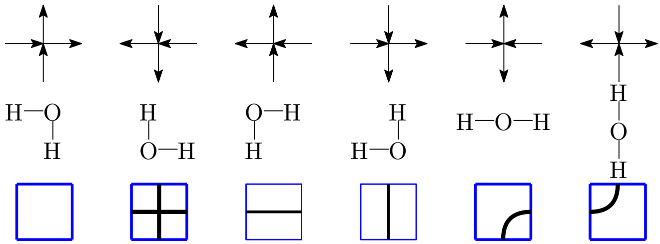
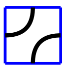


Figure 4: Top: The six vertex configurations of the six-vertex model. Center: The bijection with square ice molecules. Bottom: The corresponding bumpless pipe dream tiles.

3.2 Bumpless pipe dreams

We now transform a six-vertex configuration into another object that looks like a collection of pipes coming from a multi-level apartment building. The way to do this is to take each vertex configuration and replace it by the corresponding tile as shown in Figure 4 (bottom). These are called *bumpless pipe dreams* since a tile like this one

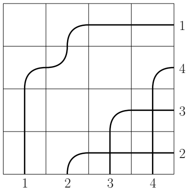


where the two pipes are nearly bumping into each other is not allowed. In Figure 5 (left), you can see the bumpless pipe dream corresponding to the square ice configuration of Figure 3. Something new you can see from this perspective is that there's a permutation associated to each ASM: Number the pipes from left to right across the bottom, then see where each pipe goes and number the boxes accordingly from top to bottom. For example, Figure 5 shows a bumpless pipe dream with pipe permutation 1432.

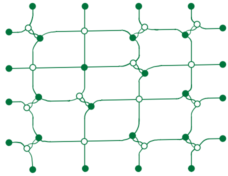
A bumpless pipe dream is *reduced* if it was made by a moderately good plumber who made sure that no pair of pipes crossed twice. Reduced bumpless pipe dreams will appear again at the end of this article as part of a partial solution to a missing bijection.

3.3 Hourglass plabic graphs

In recent work [4], alternating sign matrices showed up unexpectedly as certain graphs that are part of a larger set of objects called "hourglass plabic graphs"; see Figure 5 (right) for an example. Try to guess the rules of how to create this picture from our example ASM! Aside from creating pretty pictures, hourglass



The reduced bumpless pipe dream corresponding to the ASM from Figure 3.



The corresponding hourglass plabic graph.

Figure 5

plabic graphs help solve an algebraic problem of finding a "nice basis" for the space of "SL_4-invariant polynomials".

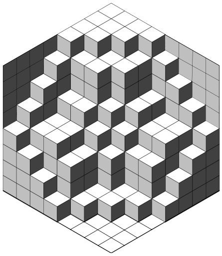
4 A missing alternating sign matrix bijection

4.1 The problem

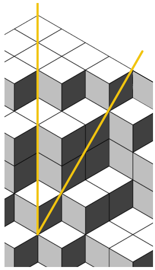
One of the most mysterious aspects of alternating sign matrices is that there are other objects with the same counting formula for which no nice bijection is known. One such object is a special type of "plane partition".

Imagine I gave you a bunch of blocks and told you to stack them in a corner of a box following certain rules: Rule 1 says that the blocks should be mirror-image symmetric to the left and right (like your face). Rule 2 says that if you pick up the box and rotate it around the corner (so that the floor becomes the left wall, the left wall becomes the right wall, and the right wall becomes the floor), the stack of blocks should look the same. Rule 3 says that the empty space in the box should be of the same shape as the stack of blocks. Mathematically, this forms a *totally symmetric self-complementary plane partition (TSSCPP)*; see Figure 6 (left). A natural question is: for a given box size, how many TSSCPPs are there? First, notice that Rule 2 implies that the lengths of the sides of the box should all be equal, meaning that the box has to be a cube. Rule 3 tells us that the space in the box needs to be evenly divided into block spaces and spaces without blocks. So the volume of the box (in block units) needs to be even. Thus, the size of the box should be $2n \times 2n \times 2n$ for some natural number n.

An amazing fact is that there are exactly as many TSSCPPs in a $2n \times 2n \times 2n$ box as there are n-by-n ASMs (proved in [2]). But unlike the objects considered in the previous section, there is no nice bijection that takes an ASM as input and outputs a TSSCPP. Of course, you could construct an uninsightful bijection in which you put all the ASMs in a line on the left side of your paper and all



A TSSCPP in an $8 \times 8 \times 8$ box.



Its fundamental domain.

Figure 6

the TSSCPPs in a line on the right side and declare that you match them up according to their order on the paper. But I hope I have convinced you by the examples of all the nice bijections in the last section that this is not really what we want. In fact, mathematicians today are still searching for a nice bijection between these two sets. In the next section, we discuss some recent progress on this open problem.

4.2 Progress

We start by finding some natural bijections on TSSCPPs. Imagine you have a TSSCPP stack of blocks in front of you and you want to tell a friend how to make the same stack without showing them a picture. Since TSSCPPs have so much symmetry, you don't actually need to tell them where each block goes in order for them to put all the blocks in the right spots. Rule 1 says that if you tell them where to put the blocks on the left side, they could put the right side blocks in the mirror-image spots. Rules 2 and 3 give more direction, so that you don't even need to tell them the locations of *all* the blocks on the left side. It turns out that you only need to know how to stack the blocks in a pizza slice-shaped area that is only $\frac{1}{12}$ of the size to have all the information needed to construct the whole thing. This slice with the essential information is called the *fundamental domain*; see Figure 6 (right). Now, when we try to find a bijection, we often start with a subset of the whole set that we understand well. A few years ago in [9], we found a simple criterion on the fundamental domain that identifies which TSSCPPs correspond to permutations! (For example, our TSSCPP from Figure 6 corresponds to the permutation 3241.) This gives a partial bijection between ASMs and TSSCPPs, as a permutation is a special case of an ASM. But what about when the ASM has (-1)'s?

More recently, we figured out a way to extend this partial bijection to a larger

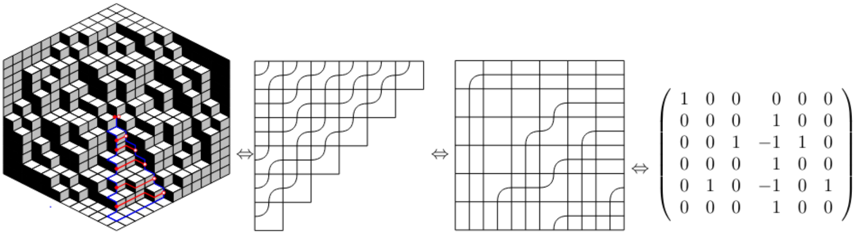


Figure 7: An example of the map from [6] between certain TSSCPPs and ASMs.

subset of ASMs, including some with (-1)'s [6]. We used a new breakthrough [5] in “Schubert calculus” which gave a bijection between reduced bumpless pipe dreams and other objects called “reduced pipe dreams”.[5] We showed how to map TSSCPPs to pipe dreams and then used the bijection from [5] to turn these pipe dreams into bumpless pipe dreams (and thus ASMs) whenever the pipe dreams are reduced. See Figure 7 for an example. This produced a much larger partial bijection between ASMs and TSSCPPs.

4.3 Outlook

Will we ever find a nice bijection between all ASMs and TSSCPPs? Mathematicians have been looking for 30 years. Some say if we haven't found it by now, it may not exist. Others say perhaps there's another way to look at the problem

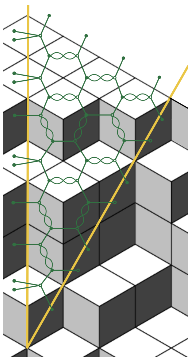


Figure 8: An hourglass plabic graph (in green) drawn on the fundamental domain of our TSSCPP.

[5] Despite their similar sounding names, these are distinct sets of objects, and the bijection between them is actually rather involved.

that will make things clearer. Time will tell, but we do have a new perspective that might help. Remember how ASMs popped up unexpectedly as hourglass plabic graphs? TSSCPPs showed up there, too (see Figure 8)! Discovering both ASMs and TSSCPPs as instances of hourglass plabic graphs does not guarantee a bijection will be found. But it's another connection that might possibly be used in the future to unlock the mystery of why they have the same counting formula.

References

[1] G. Algara-Siller, O. Lehtinen, F. C. Wang, R. R. Nair, U. Kaiser, H. A. Wu, A. K. Geim, and I. V. Grigorieva, *Square ice in graphene nanocapillaries*, Nature **519** (2015), no. 7544, 443–445.

[2] G. Andrews, *Plane partitions V: The TSSCPP conjecture*, Journal of Combinatorial Theory, Series A **66** (1994), no. 1, 28–39.

[3] D. Bressoud and J. Propp, *How the alternating sign matrix conjecture was solved*, Notices of the American Mathematical Society **46** (1999), no. 6, 637–646.

[4] C. Gaetz, O. Pechenik, S. Pfannerer, J. Striker, and J. Swanson, *Rotation-invariant web bases from hourglass plabic graphs*, Inventiones mathematicae (2025), 1–102.

[5] Y. Gao and D. Huang, *The canonical bijection between pipe dreams and bumpless pipe dreams*, International Mathematics Research Notices (2023), no. 21, 18629–18663.

[6] D. Huang and J. Striker, *A pipe dream perspective on totally symmetric self-complementary plane partitions*, Forum of Mathematics. Sigma **12** (2024), Paper No. e17, 19.

[7] G. Kuperberg, *Another proof of the alternating-sign matrix conjecture*, International Mathematics Research Notices **3** (1996), 139–150.

[8] W. H. Mills, D. P. Robbins, and H. Rumsey, *Alternating sign matrices and descending plane partitions*, Journal of Combinatorial Theory, Series A **34** (1983), no. 3, 340–359.

[9] J. Striker, *Permutation totally symmetric self-complementary plane partitions*, Annals of Combinatorics **22** (2018), no. 3, 641–671.

(Answer to the question from section 2: The middle two matrices are ASMs.)

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