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Fracture Mechanics: A Nonlocal Approach

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Nonlocal models consider interactions over a range of distances, not just at a single point. In this snapshot, we give a short introduction to nonlocal modeling, explain how it differs from its local counterpart, and present an application: fracture mechanics.

1 Introduction

> All models are wrong, but some are useful.
>
> George Box (1976)

From weather prediction and population dynamics to bridge engineering, mathematical models are used to describe our world on many levels. However, as George Box noted, no model is perfectly accurate. Choosing the right one is a balancing act: intricate models may capture many details, but are often hard to interpret. On the other hand, simpler models usually only capture the general features, but their simplicity often allows deeper insights.

For example, consider the problem of evaluating whether a bridge beam could fracture. While an intricate model might account for the number of cars, their weight, and their speed, it may ultimately prove less effective. The key information could be the existence of internal cracks in the beam or the exact

[1] M. F. and P. R. are supported by NSF award DMS-1716790.

composition and properties of the materials used. Thus, a simpler model of the spread of internal cracks could be more insightful.

The quote by George Box emphasizes the importance of a mathematical model that gives us enough accuracy while still being relatively easy to utilize. In this snapshot, we present *nonlocal models* in contrast to *local models*. While most widely used models are local, we showcase an example in which nonlocal models prove to be more accurate.

Local versus nonlocal modeling

Imagine a honeybee swarm where individuals move from one location to another without collision as is explained in [2]. The self-organization behavior naturally introduces the ideas of *nonlocality*. In particular, what one honeybee does not only influences the trajectories of the honeybees near it, but also causes a coordinated flight for the whole swarm. Similarly, in the field of dynamic fracture, it has been shown that the stresses experienced at the tip of a crack are not sufficient to determine the behavior at that point. Rather, one must collect information from a neighborhood around the tip to determine how the crack would advance.

In contrast, local models use only the information directly available at the point of interest, usually in the form of local differential operators. In materials, the stress is related to the change in displacement u within a material. Mathematically this is described by the *derivative operator*

$$\frac{\mathrm{d}u(x)}{\mathrm{d}x} = \lim_{y \to x} \frac{u(y) - u(x)}{y - x}. \tag{1}$$

The derivative operator gives the rate of increase of the function u at a specific point x. This is why we call the derivative a *local* operator.

An analogous nonlocal operator is the two-point derivative

$$\mathcal{G}u(x,y) = (u(y) - u(x))\alpha(x,y).$$

The two-point function α quantifies the interaction between two points x and y. Usually, the points nearest to the location of interest have the most influence, but that influence decreases with the distance to the reference point. Indeed, let the points at a distance greater than δ, which we call the *interaction radius*, not affect each other. A possible function α which encodes the behavior is

$$\alpha(x,y) = \begin{cases} \frac{1}{y-x}, & |y-x| < \delta \\ 0, & \text{otherwise.} \end{cases}$$

With this choice of α, the nonlocal two-point derivative is

$$\mathcal{G}u(x,y) = \begin{cases} \frac{u(y)-u(x)}{y-x}, & |y-x| < \delta \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Notice how, in contrast to the local derivative operator from Equation (1), the nonlocal derivative in Equation (2) depends on the locations of two distinct points x and y.

In the following, we focus on the systems where local information cannot describe the behavior of the system well. More specifically, we look at nonlocal modeling in fracture mechanics and how it relates to experimental data.

2 An Application: Fracture Mechanics (Peridynamics)

The most important questions in the field of fracture mechanics are when and where a crack will form, how quickly it will propagate, and if/how it will branch. The answers to these questions help us control the propagation of cracks in bridge beams, cellphone screens, and soda cans.

The local equation to model elastic deformation[2] is given by

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + b, \tag{3}$$

where u denotes displacement within the material, b is the external force, and Δ is the Laplace operator[3] $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. The Laplace operator is a local derivative operator, which makes this theory of elastic deformation local. For small displacements, such as the propagation of sound in a solid, Equation (3) agrees very well with reality.

However, near cracks in the material, the displacement changes abruptly, so derivatives are undefined – meaning that the standard equations no longer apply. In 2000, Stewart Silling reformulated the equations of fracture mechanics in terms of nonlocal operators. The theory was named *peridynamics*, originating from the Greek words *peri*, meaning “near,” and *dynamics*, meaning “force.” His first paper on the subject [3] has been cited more than 4500 times at the time of writing this snapshot. The *peridynamic equation of motion* is

$$\frac{\partial^2 u}{\partial t^2} = \mathcal{L}u + b, \tag{4}$$

where b is the external force, and

$$\mathcal{L}u(x) = \int_{|y-x|<\delta} (u(y) - u(x))\mu(x, y)\, \mathrm{d}y,$$

is the nonlocal Laplacian. The kernel of interaction $\mu(x, y)$ describes long-range interactions between points x and y. Equation (4) has the same form as

[2] We have excluded material properties for clarity.

[3] The Laplace operator can be considered as the second derivative in three spatial dimensions.

Equation (3), only the local Laplacian is replaced by its nonlocal counterpart. Indeed, if no fracture occurs and all derivatives in the material are well-defined, the nonlocal and local Laplacian of the displacement are nearly equal. This gives some agreement of the nonlocal and local theories.

Experimental data on fracture versus peridynamic simulation

After outlining the theoretical ideas behind peridynamics in fracture mechanics, we now look at how well this theory corresponds to experimental data. We look at an example from [1], where the authors compare the experimental data shown in Figure 1 from [4] to a numerical simulation using peridynamics shown in Figure 2.

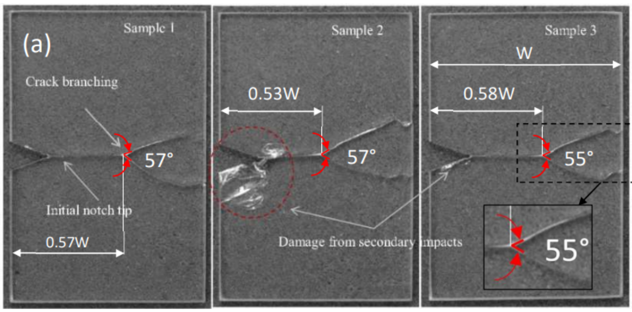

Figure 1: Three samples of crack propagation and branching within a material. Experimental data are from [4] and the measurements are added in [1].

The key observation of the experimental data in Figure 1 is that the branching occurs after propagating 50–60% of the way through the material and that the branching angle in each sample is between 55° and 57°.

Figure 2 shows the equivalent peridynamic simulation for different interaction radii δ. As in the physical samples, the crack branches after traveling 50–60% of the way through the sample. However, the branching angle in the simulation is 45°, which differs from the one measured experimentally. Notice, however, that the branching behavior of the end of the crack is similar to that observed in the experiment. Despite the quantitative difference, the qualitative match between the experiment and simulation is remarkable, and the peridynamic model notably outperforms other simulation methods considered in the paper [1].

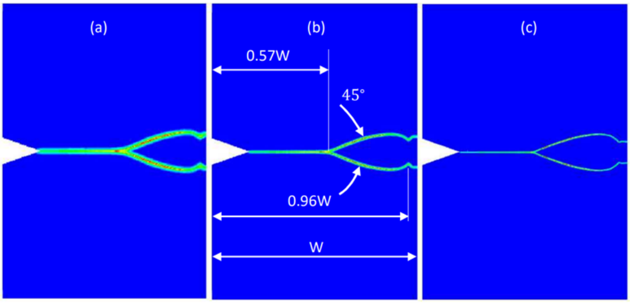

Figure 2: Crack propagation simulation obtained with a peridynamics model. Interaction radii: (a) δ=2 mm, (b) δ=1 mm, and (c) δ=0.5 mm.

Furthermore, the authors in [1] compared the experimental results from [4] of the speed of the crack tip propagation to the equivalent peridynamic simulation. Figure 3 shows the crack tip velocity starting at 0 and then shooting up to 1500 m/s, both in experiments and simulations.

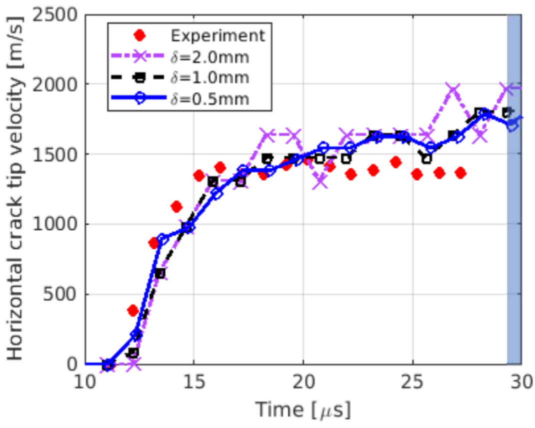

Figure 3: Crack tip velocity in experiments [4] and the peridynamics simulations [1] with interaction radii of δ = 2 mm, 1 mm, and 0.5 mm.

These results highlight how nonlocal models can capture essential features of physical systems and thus outperform their local counterparts. While quantitative differences remain, the qualitative match with experiments shows the promise of this approach.

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