 № 9/2025

Why oscillation counts: Diophantine approximation, geometry, and the Fourier transform

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Is it possible to approximate arbitrary points in space by vectors with rational coordinates, with which we, and computers, feel much more comfortable? If yes, can we approximate those points arbitrarily close? In this snapshot, we explore how the geometric configuration of these points influences the answers to these questions. Further, we delve into the closely related problem of counting rational vectors near surfaces. The unlikely tool which helps us in this endeavour is Fourier analysis – the study of waves and oscillations!

1 Introduction

For almost two millennia, humans across civilizations have asked: how well can an arbitrary real number, like π, be approximated by fractions? Indeed, this question lies at the heart of a branch of modern mathematics which is now

[1] The author is supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - EXC-2047/1 - 390685813 as well as SFB 1060.

called Diophantine approximation.[2] One of the primary applications of this area is in cryptography, particularly in the security of encryption algorithms, where the hardness of certain Diophantine problems ensures the robustness of cryptographic keys. Moreover, in the field of signal processing, it helps in minimizing errors in the representation and transformation of signals.

But what does it mean for a real number (call it α) to be *well*-approximable by fractions, also called rational numbers? Supposing you had a new classmate or coworker called α, what would it take for you to know them well? Most of us will agree that it would require contact which is both *close* and *frequent*. Indeed, the same goes for numbers too! The fundamental idea in Diophantine approximation is to understand how closely and how often α (the real number, this time) can be approximated by rationals with a given denominator.

We already know a perfect rational approximation to α when it is a rational: the number itself! We therefore consider the case when α is irrational, that means, not a fraction. In 1840, Dirichlet[3] proved that for any irrational number α and any positive integer N, there exists a fraction $\frac{p}{q}$ with $1 \leq q \leq N$ such that

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{qN} \leq \frac{1}{q^2}.$$

Since we can choose N to be as big as we please, Dirichlet's theorem tells us that α can be approximated by fractions of the form $\frac{p}{q}$ for infinitely many denominators q (in other words, *infinitely often*).

The proof of this fundamental result is small enough to fit in a paragraph (though perhaps not in the margin[4])! For a real number x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, and let

$$\{x\} = x - \lfloor x \rfloor$$

denote its *fractional part*. For example, $\{3.14\} = 3.14 - 3 = 0.14$, $\left\{\frac{22}{7}\right\} = \frac{22}{7} - 3 = \frac{1}{7}$, $\{\pi\} = \pi - 3 = 0.14159265358979323846\ldots$, and $\{3\} = 3 - 3 = 0$.

The first crucial observation is that when α is irrational, the numbers

$$\{\alpha\}, \{2\alpha\}, \ldots, \{N\alpha\}$$

are all distinct from each other. It is a fun exercise to figure out why![5] In addition, the fractional part of any number lies in the interval $[0, 1]$. Since

[2] Named after the Greek mathematician *Diophantus of Alexandria* who lived in the second century BCE.

[3] Johann Peter Gustav Lejeune Dirichlet (1805–1859) was a German mathematician, famous for fundamental contributions to analytic number theory, Fourier analysis, and mathematical physics. Among other achievements, he was one of the first to give the modern formal definition of a function.

[4] To understand the joke, see wikipedia.org/wiki/Fermat%27s_Last_Theorem.

[5] Hint: Assume $\{n\alpha\} - \{m\alpha\} = 0$ for $n \neq m$. What would this imply?

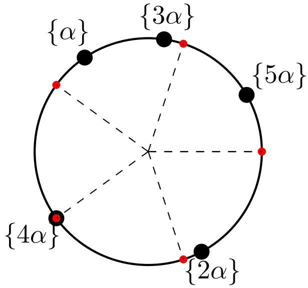


Figure 1: Dirichlet's theorem with $N = 5$ (the red dots demarcate intervals of length $\frac{1}{N}$).

further $\{0\} = \{1\} = 0$, that is, the fractional parts of both endpoints of this interval are the same, we can think of them as being the same with respect to fractional parts. Therefore, gluing the end points together transforms the interval $[0, 1]$ into a circle, which represents the fractional parts. Consequently, the numbers $\{\alpha\}, \{2\alpha\}, \ldots, \{N\alpha\}$ can be visualised as N distinct points on this circle of circumference 1 (see Figure 1).

For all of them to be accommodated, there have to be at least two which lie within a distance of $\frac{1}{N}$ from each other. In other words, we are guaranteed positive numbers $q_1 < q_2 \leq N$ such that

$$\{(q_2 - q_1)\alpha\} \leq \frac{1}{N}.$$

Let $q = q_2 - q_1$. Then $q \leq N$ and the above inequality tells us that for the integer $p = \lfloor q\alpha \rfloor$ it is true that

$$|q\alpha - p| = |\{q\alpha\}| \leq \frac{1}{N}.$$

Dividing both sides above by q then proves Dirichlet's theorem.

At this point, it is natural to wonder if it is possible for rationals to get even closer to α, infinitely often? More precisely, can we replace $\frac{1}{q^2}$ by, say, $\frac{1}{q^3}$ or more generally, $\frac{1}{q^s}$ for $s > 2$, and still conclude the above? Note that we ask for a closer approximation of α by rationals, when we increase the value of s. At some point, it simply becomes too much to ask for!

Indeed, in 1924, Khintchine[6] showed that for $s > 2$, the set of numbers α

[6] Alexandre Iakovlevitch Khintchine (1894–1959) was a Russian mathematician and one of the most significant contributors to the Soviet school of probability theory.

for which it is true that

$$\left|\alpha - \frac{p}{q}\right| \le \frac{1}{q^s} \qquad \text{for infinitely many } \frac{p}{q} \in \mathbb{Q},$$

is of negligible size. Going back to our analogy with the new classmate or coworker, this says that while it is possible to have a lot of friends, it is virtually impossible to have too many (in fact, infinitely many) very close friends!

The above result can be easily generalized to higher dimensions as well. There, the question is to approximate an n-dimensional vector $\boldsymbol{\alpha}$ by vectors with rational entries of the same denominator q. This means, the approximating vectors have entries $\frac{p_1}{q}, \dots \frac{p_n}{q} \in \mathbb{Q}$. However, the cut-off point for the exponent now is $s > \frac{1}{n} + 1$ (instead of $s > 2$ in the 1-dimensional case). More precisely, Khintchine's theorem says that for $s > \frac{1}{n} + 1$, the set of vectors $\boldsymbol{\alpha} \in \mathbb{R}^n$ which can be approximated very closely is of negligible size.

2 Counting rational points near surfaces

We now introduce some geometric flavour to our setup! Let $\mathcal{S}$ be a curve or a surface in three dimensions (the entire discussion remains valid, and even more challenging, in higher dimensions). Now $\mathcal{S}$ can be a two-dimensional surface like a plane (think of a sheet of paper, see Figure 3) or a sphere (a football); or a one-dimensional curve such as a circle (ring) or a helix (like the model of a DNA strand), see Figure 2.

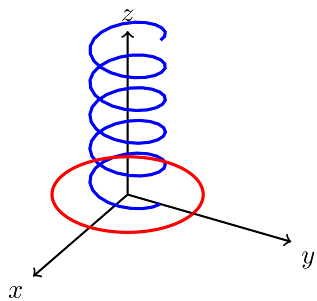


Figure 2: A helix, in blue, and a planar circle, in red.

Suppose we now want to approximate points *lying on* $\mathcal{S}$ by rational vectors. Can we prove an analogue of Khintchine's theorem for $\mathcal{S}$? In other words, can we say that for $s > \frac{1}{n} + 1$ (in our case: $n = 3$ and $s > \frac{4}{3}$), the set of points $\boldsymbol{\alpha}$ *on* $\mathcal{S}$ which can be approximated very closely, is of negligible size? Since $\mathcal{S}$ is a curve or a surface, it has no volume. Therefore, the points on $\mathcal{S}$ form a very *sparse* or *singular* collection. Consequently, approximating them with rationals is a much more delicate issue. On the contrary, the geometry of $\mathcal{S}$ itself gives us a lot of structural information about how these points are arranged in space!

The question of proving Khintchine's theorem for $\mathcal{S}$ is closely related to the problem of counting the number of three-dimensional rational points in "close proximity" to $\mathcal{S}$. Let us use the parameter δ to denote the maximal distance we allow. How small δ can be taken depends on how finely these rationals are distributed. In other words, this is the distance between two neighboring fractions in the three-dimensional lattice formed by them, see Figure 3 for an illustration of the lattice. It is determined by the size of their denominator q (the bigger it is, the finer the lattice). We now stipulate that q is bounded by some number Q.

Question 1: Let $N_{\mathcal{S}}(\delta, Q)$ denote the number of rational points of denominator q with $1 \le q \le Q$, within a distance of δ from $\mathcal{S}$, where $\delta < \frac{1}{Q}$. Can we find a formula for $N_{\mathcal{S}}(\delta, Q)$, in terms of δ and Q? If yes, then in what range of δ (in terms of Q) is the formula true?

When $\mathcal{S}$ is a planar sheet, which is, say, horizontal (parallel to the x-y-plane), the above question has a very easy (albeit uninteresting) answer. In this case, it does not matter how far or near we are to $\mathcal{S}$, it will always "catch" the maximum possible number of fractions. The geometric reason behind this is that a plane is completely flat and therefore aligns fully with the lattice formed by rational points, see Figure 3.

To work this out more precisely, we choose a specific sheet, namely

$$\mathcal{S} = \{(x, y, 0) \in \mathbb{R}^3 : x, y \in [0, 1]\}.$$

This is the square with side length 1 lying on the x-y-plane. The aim is to find for every number q with $1 \le q \le Q$ rational points $(\frac{p_1}{q}, \frac{p_2}{q}, \frac{p_3}{q})$ within a distance of δ from $\mathcal{S}$. Since $\mathcal{S}$ is parallel to the lattice of rational points, the distance from $\mathcal{S}$ is completely determined by the z-coordinate and thus given by $|\frac{p_3}{q}|$. As we are interested in $\delta < \frac{1}{q}$, also $|\frac{p_3}{q}| < \frac{1}{q}$ and hence $p_3 = 0$. Thus, all the rational points of interest lie on the plane and have coordinates $(\frac{p_1}{q}, \frac{p_2}{q}, 0)$, where both p_1 and p_2 can take any integer value between 0 and q. Therefore, for each q we obtain a possible number of $(q+1)^2$ rational points within a distance of δ from $\mathcal{S}$. Consequently,

$$N_{\mathcal{S}}(\delta, Q) \approx \sum_{q=1}^{Q} (q+1)^2 \approx Q^3.$$

However, if $\mathcal{S}$ is not completely flat, the question gets far more interesting. If the surface has random distortions, one expects the number of fractions within

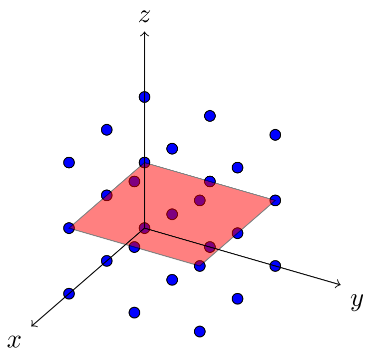


Figure 3: A planar sheet aligned parallel to a three-dimensional lattice of rational points.

a distance of δ to $\mathcal{S}$ to be proportional to an appropriate power of δ times the maximum number of such fractions allowed. This power of δ is determined by the number of constraints used to describe $\mathcal{S}$; for a two-dimensional surface (as the sheet above) this is one (the remaining dimension), while for a curve this will be two.

3 The Fourier transform enters the picture

The previous discussion tells us that the geometry of the surface $\mathcal{S}$ plays a crucial role in determining the number of rational points close to $\mathcal{S}$. However, what does it have to do with the Fourier transform?

Fourier analysis, in contrast to Diophantine approximation, is a young branch of mathematics. Its foundations were laid about two centuries ago by Fourier's[7] treatise *Théorie analytique de la chaleur* [2]. In this, he proposed that any reasonable function (say, a heat wave) can be decomposed as an infinite sum of simple sinusoidal waves of different frequencies (oscillations). This breakthrough idea was the precursor of what is now called the Fourier transform: a tool ubiquitous in mathematics, science, engineering, and information technology alike.

Why do we even care about the decomposition of a function into oscillations? This has many everyday applications outside of mathematics. Say, for example, we have a signal from a music file which we want to store economically. Then, the Fourier transform extracts all the frequencies which make the signal up.

[7] Jean-Baptiste Joseph Fourier (1768–1830) was a French mathematician, famous for initiating the investigation of Fourier series and developing the Fourier transform.

Thus, we can choose a threshold above which we decide to discard all occurring frequencies, since we know that this will not affect the signal too much. Putting the remaining frequencies together restores a signal that is very close to the original. This is the way mp3-files work. In short, the Fourier transform allows to disassemble information into components, which might be more fundamental and useful to work with.

Although the question of seeking new ways to decompose (or stitch back) a function into (or from) simpler *oscillatory* components remains fundamental, the techniques have evolved naturally into several directions. In the plane or three dimensions, the geometry and curvature of the shapes around which the frequencies of the component waves are concentrated can reveal a lot of information about a function. In such cases, the waves are often bunched together into "packets" or "tubes", and geometrical questions about how these are arranged become important.

Let w be a nice, smooth function which is zero except in a tiny region of thickness δ around the surface $\mathcal{S}$. A fundamental approach to understanding the geometry of $\mathcal{S}$ using Fourier analysis would be to study how the Fourier transform of w is concentrated in the frequency domain[8] and where it decays. In other words, we would like to see how the geometry of $\mathcal{S}$ influences the arrangement of the constituent frequencies of w.

3.1 Duality

In general, gaining a precise understanding of the decay of the Fourier transform, and linking it to the geometry of $\mathcal{S}$, is a highly delicate and complicated matter. The degree of difficulty increases as the dimension of $\mathcal{S}$ decreases; for example, the Fourier transform of a function which lives near one-dimensional curves can be incredibly complicated!

However, at least when $\mathcal{S}$ is a two-dimensional surface in space, we have more tools at our disposal. As the planar example in the beginning already showed, complete flatness prevents interesting things from happening. Thus, it is more promising to deal with surfaces with curvature. The perfect example is that of a sphere, though some might also prefer the two-dimensional paraboloid (see Figure 4).

Suppose that $\mathcal{S}$ is a paraboloidal surface as in Figure 4, which has no flat parts. If the function w lives in a small neighborhood of $\mathcal{S}$, it turns out that its Fourier transform is also concentrated around another two-dimensional surface $\mathcal{S}^*$ in the frequency domain, which we call the *dual* of $\mathcal{S}$. Moreover (and it takes some work to see this), the surface $\mathcal{S}^*$ also has no flat parts. In other

[8] This is the space where the Fourier transform of a function lives. The name is motivated from the idea to disassemble an oscillation into its constituting frequencies.

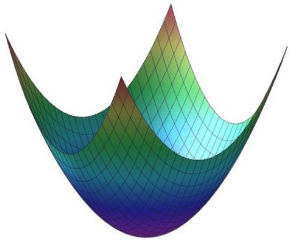


Figure 4: The paraboloid which is described by the equation $z = x^2 + y^2$.

words, $\mathcal{S}$ and $\mathcal{S}^*$ essentially turn out to be geometric twins!

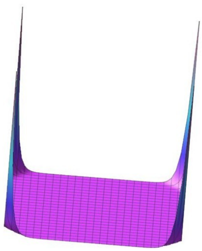
On the contrary, what if $\mathcal{S}$ does have points around which it is "flat" to a certain degree? Things get even more interesting in such situations! If $\mathcal{S}$ is a two-dimensional, *locally flat* surface, its dual $\mathcal{S}^*$ is a *locally rough* two dimensional surface, with spikes at isolated points! Consider the example illustrated in Figure 5.

3.2 Exploiting oscillation

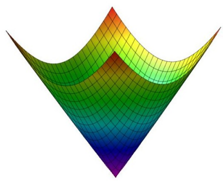
Using the Fourier transform converts the counting problem around $\mathcal{S}$ to a dual counting problem around $\mathcal{S}^*$. Moreover, fortunately for us, our notion of duality turns out to have a great property: taking duals twice brings us back to the original surface

$$(\mathcal{S}^*)^* = \mathcal{S}.$$

However, is this repeated switching between the original and dual counting problems getting us any closer to obtaining a good estimate for the counting



(a) The surface which is described by the equation $z = \left(x^2 + y^2\right)^{20}$.



(b) Its dual surface.

Figure 5: A locally flat surface $\mathcal{S}$ and its locally "spiky" dual $\mathcal{S}^*$.

function associated to $\mathcal{S}$? This incessant ping-pong would be of no benefit to us unless something significant was happening at each step to give us some leverage! This comes in the form of a powerful tool from Fourier analysis called the method of stationary phase. This method gives a precise description of the Fourier transform of functions living on surfaces without flat parts; in particular, its decay (or lack of) in different directions. For surfaces with flat parts, we need an even more elaborate analysis.

We exploit this information about the decay of the Fourier transform using a fundamental tool in analytic number theory; the *Poisson summation formula* equates evaluations of a function at discrete points of a periodic lattice to discrete evaluations of its Fourier transform. A refined geometric knowledge about the decay of the Fourier transform of functions which live on nice surfaces enables us to obtain the desired upper bounds on the number of rational points in tiny regions around these shapes. The set of well-approximable points on these surfaces, by their very definition, are contained in infinitely many such balls around fractions with shrinking radii. Thus, obtaining a tight bound on the *number* of these fractions, and therefore balls, enables us to estimate the volume of the set of these approximable points. If the number of fractions is small enough, in the limiting case, we can show that the volume of the intersection of these balls has to be negligible.

We desist from going into further details as it is beyond the scope of this snapshot. However, for the interested reader, we end our write-up with a brief description of a few older and recent results in this area, which has seen some exciting developments in the last few years! The list is by no means exhaustive.

4 State of the art and further reading

The random model for the number of rational points within a certain distance of $\mathcal{S}$, as described in Section 2, was proven to be correct for planar curves with non-vanishing curvature in the seminal works [4, 9]. The influential paper [3] established the same for surfaces of dimension one less than the ambient space and with non-vanishing curvature (like spheres). More recently, in [8], N. Technau and the author showed that the random model is true even for a special class of surfaces where the curvature vanishes at exactly one point (or a few isolated points).

The breakthrough paper [1] published last year established the convergence case of Khintchine's theorem for surfaces of arbitrary dimension under a mild curvature condition. Shortly afterwards, in the joint work [7] with D. Schindler and N. Technau, the author developed a novel combination of Fourier analytic methods and deep results from homogeneous dynamics, to make new progress on answering Question 1 for these general surfaces.

Finally, for an introduction to related ideas from Fourier analysis and Diophantine approximation, we invite the reader to consult Snapshots 6/2023 [5], 1/2024 [10], 6/2020 [11], and 3/2019 [6].

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Mathematical subjects
Algebra and Number Theory, Analysis

Connections to other fields
Computer Science, Engineering and Technology, Physics

DOI
10.14760/SNAP-2025-009-EN

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ISSN 2626-1995

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