Snapshots of modern mathematics
from Oberwolfach

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Is there a smooth lattice polytope which does not have the integer decomposition property?

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We introduce Tadao Oda's famous question on lattice polytopes which was originally posed at Oberwolfach in 1997 and, although simple to state, has remained unanswered. The question is motivated by a discussion of the two-dimensional case – including a proof of Pick's Theorem, which elegantly relates the area of a lattice polygon to the number of lattice points it contains in its interior and on its boundary.

1 Introduction

Lattice polytopes are fundamental objects in mathematics and play a crucial role in a broad range of subjects such as discrete and algebraic geometry, algebra, combinatorics, coding theory, and optimisation theory. They arise naturally in a variety of unexpected or even surprising ways. Consider the following classical question from enumerative combinatorics, for instance.

Question 1. How many monomials in three variables of a given degree m are there?

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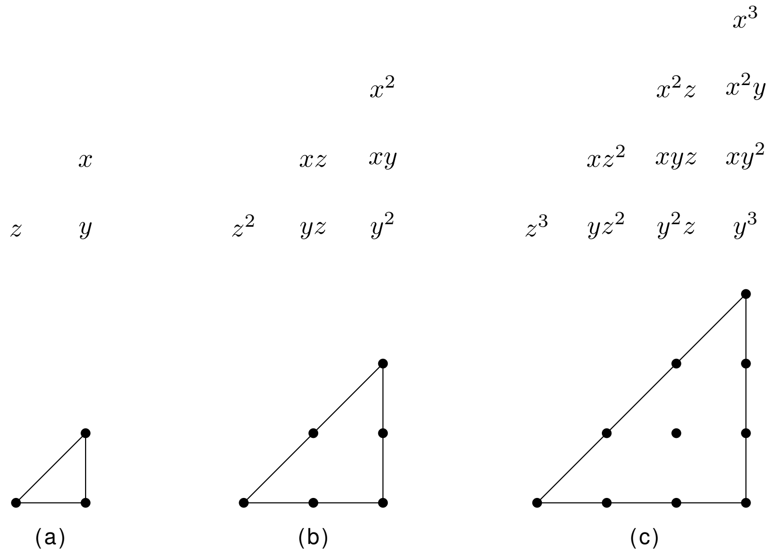

Figure 1: Interpreting monomials in three variables x, y, and z as lattice points in a triangle, for (a) degree 1, (b) degree 2, and (c) degree 3.

A *monomial* is a product of (powers of) certain variables; its *degree* is the sum of the exponents as they appear in the product. We thus wish to find the total number of all triples of non-negative integers whose sum equals a given integer m.

Let x, y, and z be the three variables in Question 1. We consider some examples first, and we obtain three monomials of degree 1, six monomials of degree 2, and ten monomials of degree 3; see Figure 1. In the process, we see that the monomials in x, y, and z of degree m can be arranged in a triangular shape where the exponent of x decreases from the top down, and the exponent of z decreases from the left to the right. The exponent of the remaining variable y is completely determined by the two other exponents and the total degree m. We conclude that the number of monomials in three variables of a given degree equals the number of points in a certain triangle, as illustrated by Figure 1.

More precisely, the number of monomials in x, y, and z of degree m equals the number of *lattice points* – that is, points with integer-valued coordinates – in the triangle with vertices at $(0,0)$, $(m,0)$, and (m,m). In other words, let T be the triangle with the vertices $(0,0)$, $(1,0)$, and $(1,1)$, let $mT = \{m\mathbf{v} \mid \mathbf{v} \in T\}$ be the *m-th dilation* of T, and let $L_T(m) = |mT \cap \mathbb{Z}^2|$ count the number of lattice points in mT. Then, the number of monomials in three variables of degree m equals $L_T(m)$.

In order to answer Question 1, we thus wish to find a formula for $L_T(m)$. The set of all lattice points contained in the triangle mT can be constructed by removing the diagonal from a square of side length $m+1$ and, subsequently, discarding one of the two resulting congruent triangles.[3] From this construction, it can be seen that a formula as desired is given by

$$L_T(m) = \frac{(m+2)^2 - (m+2)}{2} = \frac{1}{2}m^2 + \frac{3}{2}m + 1\,.$$

Notice that the leading coefficient coincides with the area of T. This is not a coincidence but part of a bigger story known as *Ehrhart theory.*

Our goal is to introduce a famous question asked by Tadao Oda at Oberwolfach in 1997. Roughly speaking, Oda wondered whether the lattice points within a polytope, given that it is of a certain type, always satisfy an elegant counting property; see Question 2 at the end of this snapshot for a precise statement. It will become apparent that a certain instance of this problem is related to Question 1 in that, in view of how we counted the monomials, it amounts to the question whether each monomial of degree m can be written as a product of m variables. Of course, one immediately sees that this is the case. However, Oda's question is not so easy in full generality; although simple to state, it remains unanswered.

Let us now introduce the general picture. We work with the lattice of integral points $\mathbb{Z}^d \subset \mathbb{R}^d$, that is, the set which consists of all points in d-dimensional space whose coordinates are integers. A subset $C \subset \mathbb{R}^d$ is called *convex* if every straight line segment which connects two points in C lies entirely within C. The *convex hull* of a set of points $B \subset \mathbb{R}^d$ is the inclusion-wise smallest subset $\operatorname{conv}(B) \subset \mathbb{R}^d$ which is convex and contains B. For $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, an alternative description is as follows:

$$\operatorname{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \;\middle|\; \lambda_1, \ldots, \lambda_n \in [0,1], \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

The term *lattice polytope* shall describe the convex hull $P = \operatorname{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ of a finite number of lattice points $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

As before, the m-th dilation of P is the polytope $mP = \{m\mathbf{v} \mid \mathbf{v} \in P\}$. The value of the *Ehrhart function* L_P at $m \in \mathbb{Z}_{\geq 0}$ is defined to be the number of lattice points $L_P(m) = |mP \cap \mathbb{Z}^d|$ in mP. The *dimension* of P, denoted by $\dim(P)$, is the dimension of the smallest affine subspace containing P.[4]

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[3] Note that a square with side length $m+1$ contains $(m+2)^2$ lattice points. The diagonal contains $m+2$ lattice points.

[4] The term *affine subspace* refers to a point, line, plane, … in $\mathbb{R}^d$. For example, a non-degenerate triangle is contained in a plane but in no line; therefore, such a triangle is of dimension 2.

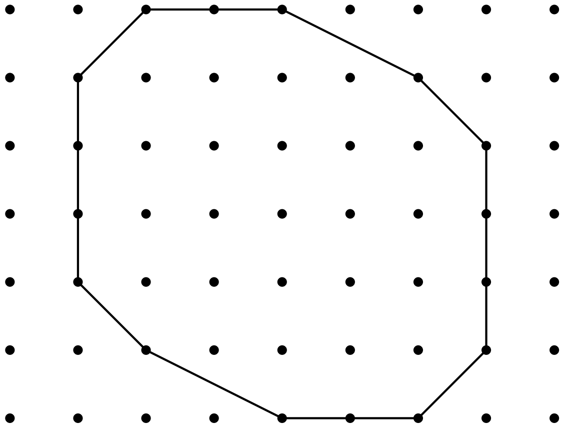

Figure 2: Computing the area of a lattice polygon using Pick's Theorem. There are 23 interior lattice points and 16 boundary lattice points.

Theorem 1 (see [3, 4]). *There exists a polynomial f of degree* $\dim(P)$ *with rational coefficients such that $L_P(m) = f(m)$ for all $m \in \mathbb{Z}_{\geq 0}$. Furthermore, the leading coefficient of f coincides with the Euclidean volume* $\mathrm{vol}(P)$ *of P.*

Theorem 1 allows us to interpret L_P as a polynomial of degree $\dim(P)$ which we call the *Ehrhart polynomial* of P. That the leading coefficient coincides with the volume of the polytope P is surprising. However, if P is a *polygon*, that is, if P is a polytope of dimension 2, then the relationship between the area of P and the lattice points it contains can be made even more precise.

Theorem 2 (Pick's Theorem). *For a lattice polygon $P \subset \mathbb{R}^2$, the Euclidean volume* $\mathrm{vol}(P)$ *is given by the formula*

$$\mathrm{vol}(P) = |P^\circ \cap \mathbb{Z}^2| + \frac{|\partial P \cap \mathbb{Z}^2|}{2} - 1\,.$$

Here, $|P^\circ \cap \mathbb{Z}^2|$ denotes the number of interior lattice points, and $|\partial P \cap \mathbb{Z}^2|$ denotes the number of boundary lattice points of P.

Let us use Pick's Theorem to compute the area of the polygon P shown in Figure 2. It can be seen that P has 23 interior lattice points and 16 boundary lattice points, so its area is given by

$$\mathrm{vol}(P) = 23 + \frac{16}{2} - 1 = 30\,.$$

2 Proof of Pick's Theorem

The proof of Pick's Theorem contains numerous beautiful ideas and constructions from Ehrhart theory. Since several of these play an important role in motivating Oda's Oberwolfach question, we give a proof here. The fundamental idea is to proceed by induction on the number of lattice points $|P \cap \mathbb{Z}^2|$.[5]

2.1 The base case

We assume that P is a triangle whose vertices $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are its only lattice points; that is, $|P \cap \mathbb{Z}^2| = 3$. Such triangles are called *empty*. We will show that, in this case, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ constitute an *affine basis* of $\mathbb{Z}^2$; that is, the difference vectors $\mathbf{v}_1 - \mathbf{v}_3$ and $\mathbf{v}_2 - \mathbf{v}_3$ form a *lattice basis* of $\mathbb{Z}^2$.[6] Let us assume this for a moment and show how it implies Pick's Theorem in the base case.

By assumption, $\mathbf{v}_1 - \mathbf{v}_3$ and $\mathbf{v}_2 - \mathbf{v}_3$ form a lattice basis of $\mathbb{Z}^2$. The 2×2-matrix A whose rows are these difference vectors is invertible over $\mathbb{Z}$. In other words, there exists another 2×2-matrix B with entries in $\mathbb{Z}$ such that the matrix product AB equals the identity matrix, and thus the determinant[7] of A must itself be invertible over $\mathbb{Z}$; that is, the determinant of A is 1 or -1. It is a geometrical fact that the absolute value of the determinant $\det\binom{\mathbf{a}}{\mathbf{b}}$ of the 2×2-matrix with rows $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^2$ equals the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$. Hence, the area of P equals $1/2|\det(A)| = 1/2 = 0 + 3/2 - 1$, as desired.

It remains to show that $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ together form an affine basis of $\mathbb{Z}^2$. Let $\mathcal{P}$ denote the parallelogram spanned by $\mathbf{v}_1 - \mathbf{v}_3$ and $\mathbf{v}_2 - \mathbf{v}_3$; that is,

$$\mathcal{P} = \{\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 - (\lambda_1 + \lambda_2)\mathbf{v}_3 \mid 0 \leq \lambda_i \leq 1\}.$$

Such a parallelogram is shown in Figure 3. We claim that the only lattice points contained in $\mathcal{P}$ are its vertices $\mathbf{0}, \mathbf{v}_1 - \mathbf{v}_3, \mathbf{v}_2 - \mathbf{v}_3$, and $\mathbf{v}_1 + \mathbf{v}_2 - 2\mathbf{v}_3$. Indeed, the triangle with the vertices $\mathbf{0}$, $\mathbf{v}_1 - \mathbf{v}_3$, and $\mathbf{v}_2 - \mathbf{v}_3$ is empty, and so its opposite triangle, that is, the triangle with the vertices $\mathbf{v}_1 - \mathbf{v}_3$, $\mathbf{v}_2 - \mathbf{v}_3$, and $\mathbf{v}_1 + \mathbf{v}_2 - 2\mathbf{v}_3$ which covers the other half of $\mathcal{P}$, is also empty. Certainly,

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[5] This means that the theorem is first proved for polygons containing exactly three lattice points (Section 2.1). In a second step, it is proved that if the statement of the theorem holds for all polygons which contain at most N lattice points, then it also holds for all polygons containing $N + 1$ lattice points, where $N \geq 3$ is any integer (Section 2.2). As a consequence, the theorem must then hold for all integers $N \geq 3$.

[6] The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_d \in \mathbb{Z}^d$ are said to form a *lattice basis* if each vector $\mathbf{v} \in \mathbb{Z}^d$ admits a unique representation of the form $\mathbf{v} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_d \mathbf{v}_d$ for $\lambda_1, \ldots, \lambda_d \in \mathbb{Z}$; the $d + 1$ elements $\mathbf{v}_1, \ldots, \mathbf{v}_{d+1} \in \mathbb{Z}^d$ are said to form an *affine basis* if the difference vectors $\mathbf{v}_1 - \mathbf{v}_{d+1}, \ldots, \mathbf{v}_d - \mathbf{v}_{d+1}$ form a lattice basis.

[7] The value of the determinant of a 2×2-matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by the expression $ad - bc$.

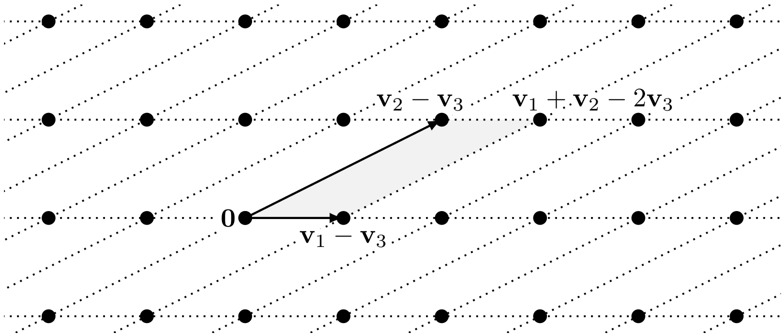

Figure 3: Tiling the plane with translations of a parallelogram which comes from an empty triangle with the vertices $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$.

the set $\{\mathcal{P} + \mathbf{v} \mid \mathbf{v} \in \mathbb{Z}^2\}$ of all translations of $\mathcal{P}$ tiles the plane, as illustrated in Figure 3, from which we may observe that every lattice point in $\mathbb{Z}^2$ is a vertex of a lattice translation of $\mathcal{P}$. Hence, each $\mathbf{x} \in \mathbb{Z}^2$ can be expressed as a linear combination $\mathbf{x} = k_1(\mathbf{v}_1 - \mathbf{v}_3) + k_2(\mathbf{v}_2 - \mathbf{v}_3)$ for $k_1, k_2 \in \mathbb{Z}$. Moreover, this representation is unique, so $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ together form an affine basis of $\mathbb{Z}^2$.

2.2 The inductive step

Let us assume Pick's Theorem is valid for all lattice polygons $Q \subset \mathbb{R}^2$ with $|Q \cap \mathbb{Z}^2| \leq N$ for some $N \geq 3$. We need to show that it then holds for all lattice polygons P with $|P \cap \mathbb{Z}^2| = N + 1$. Let P be such a polygon. The inductive step splits into two cases: either there are exactly three lattice points on the boundary ∂P of P, or $|\partial P \cap \mathbb{Z}^2| > 3$.

If $|\partial P \cap \mathbb{Z}^2| = 3$, then the interior P° of P contains at least one lattice point $\mathbf{v}$. The three line segments connecting $\mathbf{v}$ with the three vertices of P split the polygon into three subpolygons P_1, P_2, and P_3, as illustrated by Figure 4. We shall count the lattice points in each of the P_k, $k = 1, 2, 3$, and relate these counts to the number of lattice points in P. Let $i_k = |P_k^\circ \cap \mathbb{Z}^2|$, and let $b_k = |\partial P_k \cap P^\circ \cap \mathbb{Z}^2| = |\partial P_k \cap \mathbb{Z}^2| - 2$, for $k = 1, 2, 3$.

Since $|P_k \cap \mathbb{Z}^2| \leq N$ for $k = 1, 2, 3$, we can apply our initial assumption on the validity of Pick's Theorem for lattice polygons containing at most N lattice points to each of the polygons P_1, P_2, P_3, and we obtain

$$\begin{aligned} \operatorname{vol}(P_k) &= |P_k^\circ \cap \mathbb{Z}^2| + \frac{|\partial P_k \cap \mathbb{Z}^2|}{2} - 1 \\ &= i_k + \frac{b_k + 2}{2} - 1\,. \end{aligned}$$

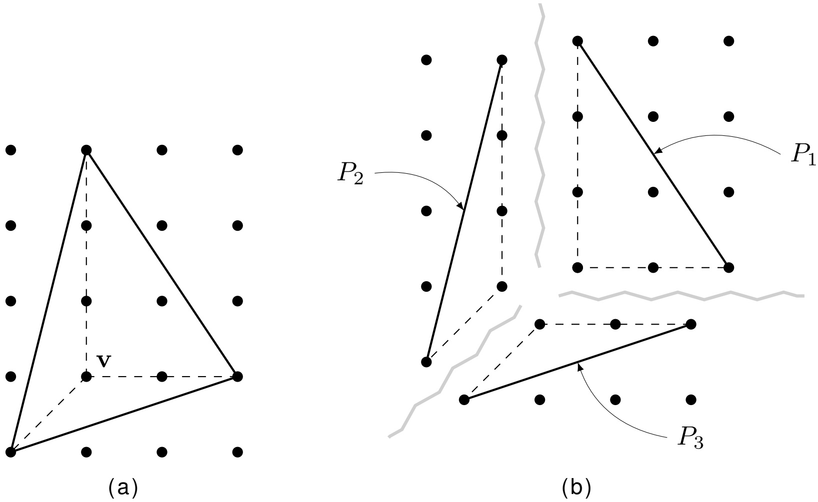

Figure 4: A lattice polygon (a) with exactly three boundary lattice points is cut (b) into three subpolygons P_1, P_2, and P_3. Here, $b_1 = 4$, $b_2 = 3$, $b_3 = 2$, $i_1 = 1$, and $i_2 = i_3 = 0$.

The area of P equals the sum of the areas of the subpolygons P_1, P_2, and P_3. By further counting the lattice points in each of the three subpolygons separately, we arrive at the following chain of equalities:

$$\begin{aligned}
\operatorname{vol}(P) &= \sum_{k=1}^{3} \operatorname{vol}(P_k) \\
&= \sum_{k=1}^{3} \left(i_k + \frac{b_k + 2}{2} - 1 \right) \\
&= \sum_{k=1}^{3} i_k + \frac{\sum_{k=1}^{3} b_k}{2} \\
&= |P^\circ \cap \mathbb{Z}^2| + \frac{1}{2} \\
&= |P^\circ \cap \mathbb{Z}^2| + \frac{3}{2} - 1 \\
&= |P^\circ \cap \mathbb{Z}^2| + \frac{|\partial P \cap \mathbb{Z}^2|}{2} - 1\,.
\end{aligned}$$

Notice that each lattice point which lies on the interior parts of the dissecting lines is counted twice by the term $\sum_{k=1}^{3} b_k$, except for $\mathbf{v}$, which is counted three times. This is accounted for by the addition of $1/2$ in the fourth line.

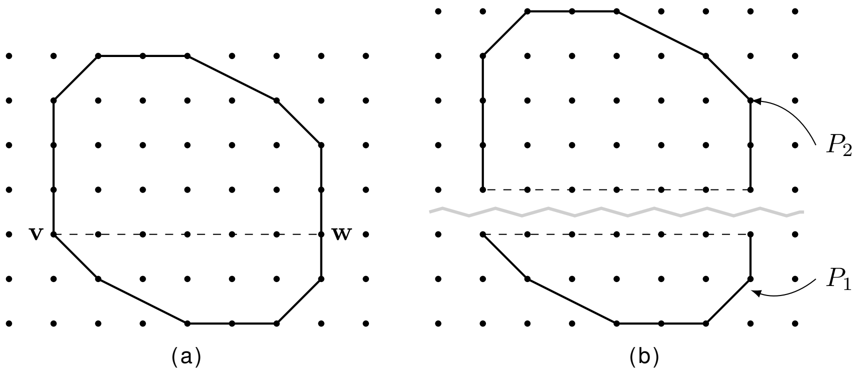

Figure 5: A lattice polygon (a) with four or more lattice points on its boundary can be split (b) into two subpolygons P_1 and P_2. Here, $i_1 = 4$, $i_2 = 14$, $b_1 = 7$, $b_2 = 11$, and $i = 5$.

If $|\partial P \cap \mathbb{Z}^2| > 3$, we can find $\mathbf{v}, \mathbf{w} \in \partial P \cap \mathbb{Z}^2$ such that the line segment connecting $\mathbf{v}$ and $\mathbf{w}$ divides P into two subpolygons P_1 and P_2 with $|P_k \cap \mathbb{Z}^2| \leq N$ for $k = 1, 2$. This is illustrated by Figure 5. We again count the number of lattice points in the subpolygons P_1 and P_2, and we relate these counts to the number of lattice points in P. Let $i_k = |P_k^\circ \cap \mathbb{Z}^2|$, and let $b_k = |(\partial P_k \setminus P^\circ) \cap \mathbb{Z}^2|$ be the number of boundary lattice points of P_k that are not contained in the interior of P. By i we denote the number of interior lattice points in the line segment from $\mathbf{v}$ to $\mathbf{w}$. Since $|P_k \cap \mathbb{Z}^2| \leq N$, we can apply the inductive hypothesis to obtain

$$\operatorname{vol}(P_k) = i_k + \frac{b_k + i}{2} - 1\,.$$

The area of P equals the sum of the areas of P_1 and P_2; hence,

$$\begin{aligned}
\operatorname{vol}(P) &= \operatorname{vol}(P_1) + \operatorname{vol}(P_2) \\
&= \left(i_1 + \frac{b_1 + i}{2} - 1\right) + \left(i_2 + \frac{b_2 + i}{2} - 1\right) \\
&= (i_1 + i_2 + i) + \frac{b_1 + b_2}{2} - 2 \\
&= |P^\circ \cap \mathbb{Z}^2| + \frac{|\partial P \cap \mathbb{Z}^2|}{2} - 1\,.
\end{aligned}$$

Regarding the final equality, notice that $i_1 + i_2 + i = |P^\circ \cap \mathbb{Z}^2|$, whereas $b_1 + b_2$ counts the lattice points on the boundary of P, counting $\mathbf{v}$ and $\mathbf{w}$ twice, however.

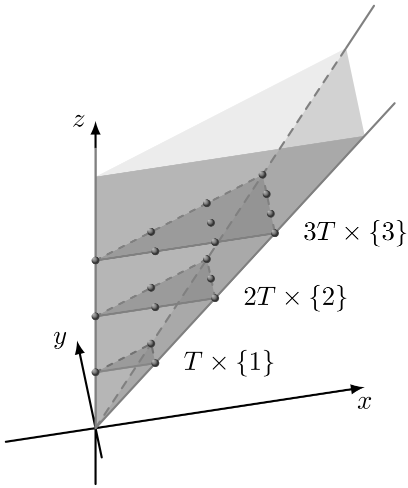

Figure 6: The cone C_T over the empty triangle T, affinely embedded into $\mathbb{R}^3$.

3 Oda's Oberwolfach question

Recall our reinterpretation of Question 1 as a problem on the number of lattice points in dilations of the empty triangle T with vertices at $(0,0)$, $(1,0)$, and $(1,1)$; the lattice points contained in the set $T \cap \mathbb{Z}^2$ each correspond to a monomial of degree 1. We now construct a single combinatorial object which simultaneously encodes the lattice points (and hence their counts) in all dilations mT, $m \in \mathbb{Z}_{\geq 0}$.

We begin by identifying $\mathbb{R}^2$ with the subset $\mathbb{R}^2 \times \{1\}$ of $\mathbb{R}^3$; in other words, we think of the plane as lying inside 3-dimensional space at height 1. In this copy of $\mathbb{R}^2$, we consider the triangle T. The *cone over* T is defined as $C_T = \{\lambda(\mathbf{x}, 1) \mid \lambda \in \mathbb{R}_{\geq 0}, \mathbf{x} \in T\}$; it is depicted in Figure 6. We may think of C_T as the union of all half-lines that start at the origin and pass through a point in $T \times \{1\}$. The cross-section of C_T at height $m \in \mathbb{Z}_{\geq 0}$, or, more precisely, $C_T \cap \{(x,y,z) \mid z = m\} = (mT, m)$, can be identified with the dilation mT of T. Hence, the cone C_T encodes the lattice points in all dilations of T.

This construction can be generalised to arbitrary lattice polytopes $P \subset \mathbb{R}^d$. Indeed, the *cone over* P can be defined as

$$\begin{aligned} C_P &= \mathbb{R}_{\geq 0}(P \times \{1\}) \\ &= \{\lambda(\mathbf{x}, 1) \mid \lambda \in \mathbb{R}_{\geq 0}, \mathbf{x} \in P\}\,. \end{aligned}$$

As before, the cross-section of C_P at height $m \in \mathbb{Z}_{\geq 0}$ can be identified with the dilation mP.

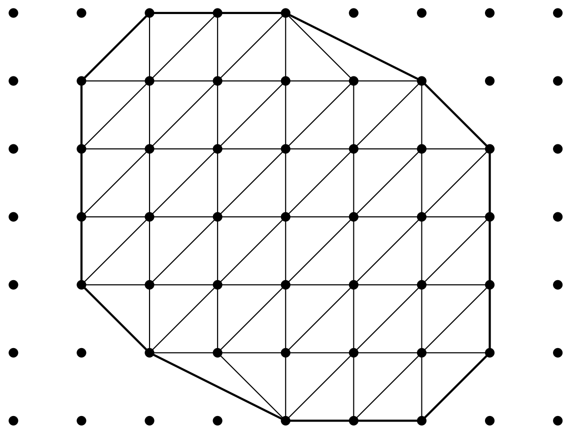

Figure 7: Every two-dimensional polygon can be covered by empty triangles.

3.1 The integer decomposition property

A lattice polytope $P \subset \mathbb{R}^d$ is said to have the *integer decomposition property* if each lattice point in C_P at height m can be written as the sum of m (not necessarily distinct) lattice points at height 1.

Example 1. If the polytope under consideration is an empty triangle T, this means that for all $(\mathbf{x}, m) \in C_T \cap \mathbb{Z}^3$ there exist (not necessarily distinct) points $(\mathbf{x}_1, 1), \ldots, (\mathbf{x}_m, 1) \in C_T \cap \mathbb{Z}^3$ such that $(\mathbf{x}, m) = (\mathbf{x}_1, 1) + \cdots + (\mathbf{x}_m, 1)$. Indeed, every empty triangle T has the integer decomposition property; the interested reader is invited to think about why this is the case.

It is natural to ask which other lattice polytopes have the integer decomposition property. Let us study some important classes of lattice polytopes for which this question has been resolved.

Example 2. We generalise empty lattice triangles to arbitrary dimension. If $\mathbf{v}_0, \ldots, \mathbf{v}_d$ form an affine basis of $\mathbb{Z}^d$, their convex hull $S = \operatorname{conv}(\mathbf{v}_0, \ldots, \mathbf{v}_d)$ is called a *d-dimensional unimodular simplex.* We claim that such a simplex S has the integer decomposition property. Indeed, since $\mathbf{v}_0, \ldots, \mathbf{v}_d$ is an affine basis of $\mathbb{Z}^d$, it follows that $(\mathbf{v}_0, 1), \ldots, (\mathbf{v}_d, 1)$ is a lattice basis of $\mathbb{Z}^{d+1}$. Hence, every lattice point $(\mathbf{x}, m)$ in the cone C_S can be expressed as a linear combination $(\mathbf{x}, m) = \sum_{i=1}^{d} \lambda_i(\mathbf{v}_i, 1)$, where $\lambda_i \in \mathbb{Z}$. Since $(\mathbf{x}, m)$ is in C_S, it follows that $\lambda_i \geq 0$, and thus S has the integer decomposition property.

Example 3. Every two-dimensional lattice polygon can be covered by empty lattice triangles; Figure 7 provides an example of how this can be achieved.

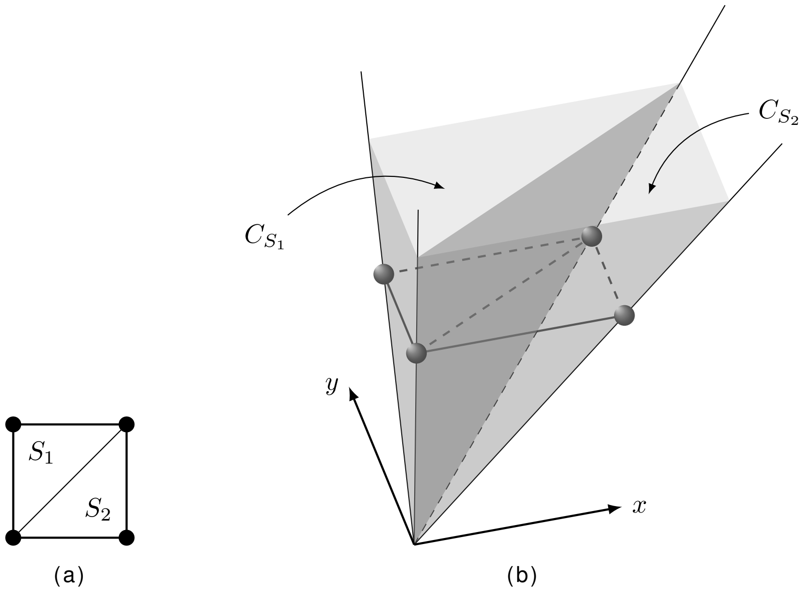

Figure 8: If (a) triangles S_i cover a polygon P, then (b) the cones C_{S_i} cover C_P.

More generally, suppose P is a lattice polytope in $\mathbb{R}^d$ which is covered by unimodular simplices $S_1, \dots, S_n$; that is, $P = S_1 \cup \cdots \cup S_n$. Such a covering is called a *unimodular covering* of P. The cone C_P over P is covered by the cones $C_{S_1}, \dots, C_{S_n}$ over the simplices S_i; Figure 8 illustrates this for the case $d = 2$. Hence, every lattice point in C_P at height m can be written as a sum of m lattice points at height 1, namely those which are located at height 1 in the corresponding cone C_{S_i}. In particular, all two-dimensional lattice polygons have the integer decomposition property, since they admit a unimodular covering. In higher dimensions, it is typically very hard to determine whether a unimodular covering even exists.[8]

Example 4. Reeve [10] describes an infinite family of tetrahedra whose members we now call the *Reeve tetrahedra*; see Figure 9. Let

$$R_r = \operatorname{conv}((0,0,0), (1,0,0), (0,1,0), (1,1,r))$$

[8] The unimodular coverings in Figures 7 and 8 have the property that the simplices intersect along common faces. Such coverings are called *triangulations*. In dimension 2, a unimodular triangulation always exists. However, there exist higher-dimensional lattice polytopes that admit a unimodular covering but no unimodular triangulation [5, Example 10].

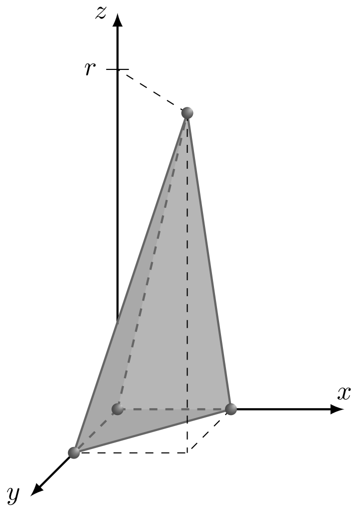

Figure 9: The Reeve tetrahedron R_r.

for some $r \in \mathbb{Z}_{\geq 1}$. The Reeve tetrahedron R_r contains exactly four lattice points, has no interior lattice points, and $\mathrm{vol}(R_r) = {}^{r}/_{6}$. Moreover, the number of lattice points in its m-th dilation is given by

$$L_{R_r}(m) = \frac{r}{6}m^3 + m^2 + \left(2 - \frac{r}{6}\right)m + 1\,.$$

It is readily seen that R_r does not admit a unimodular covering when $r > 1$, as its four vertices do not form an affine lattice basis in this case. This stands in contrast with the two-dimensional case, where every empty triangle is unimodular. The existence of empty simplices whose vertices do not form an affine basis is what makes the study of higher-dimensional lattice polytopes much richer – and harder – than that of polygons. In particular, we should not expect a direct analogue of Pick's Theorem in dimension 3 or more.

We now show that R_r does not satisfy the integer decomposition property. Let C_r be the cone over R_r. The set of lattice points $C_r \cap \mathbb{Z}^4$ forms a semigroup with $\mathbb{Z}_{\geq 0}$-basis given by

$$\begin{cases} (0,0,0,1),(1,0,0,1),(0,1,0,1),(1,1,1,1) & \text{if } r = 1, \text{ or} \\ (0,0,0,1),(1,0,0,1),(0,1,0,1),(1,1,r,1), \\ (1,1,1,2),\ldots,(1,1,r-1,2) & \text{if } r > 1. \end{cases}$$

In particular, for $r > 1$ the point $(1,1,1,2) \in C_r$ at height 2 cannot be written as the sum of two points at height 1.

3.2 The smooth case

Let $\mathbf{e}_i \in \mathbb{R}^d$ denote the i-th standard basis vector, and let us write

$$\operatorname{cone}(X) = \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{x}_i \;\middle|\; n \in \mathbb{Z}_{\geq 0}, \lambda_i \in \mathbb{R}_{\geq 0}, \mathbf{x}_i \in X \right\}$$

for the convex cone generated by a (not necessarily finite) subset $X \subset \mathbb{R}^d$. A cone $C \subset \mathbb{R}^d$ is called *smooth* if it can be identified with $\operatorname{cone}(\mathbf{e}_1, \ldots, \mathbf{e}_d)$ via a change of basis of $\mathbb{Z}^d$. In other words, it is called smooth if there exists a lattice basis $\mathbf{b}_1, \ldots, \mathbf{b}_d$ of $\mathbb{Z}^d$ with the property that the map $\mathbb{Z}^d \to \mathbb{Z}^d$ sending $(x_1, \ldots, x_d) \mapsto \sum_{k=1}^{d} x_k \mathbf{b}_k \in \mathbb{Z}^d$ is invertible and sends $\operatorname{cone}(\mathbf{e}_1, \ldots, \mathbf{e}_d)$ to C.

A lattice polytope $P \subset \mathbb{R}^d$ is called *smooth* if $\operatorname{cone}(P - \mathbf{v})$ is smooth for every vertex $\mathbf{v}$ of P. Oda's Oberwolfach question now asks whether smoothness is a sufficient condition for a lattice polytope to have the integer decomposition property.

Question 2 (Oda's Oberwolfach question [9])**.** Does every smooth lattice polytope have the integer decomposition property?

An indication that this may indeed be the case lies in the fact that the smoothness property of a polytope P ensures that each of its corners is covered by a dilation of a unimodular simplex, and in the speculation that it should be possible to extend these corner covers sufficiently far inside so as to yield a unimodular covering of the polytope P.

Despite many efforts to answer Question 2, it is still wide open. Moreover, there is currently no general consensus on the likely answer. Substantial efforts have been made in order to find a counterexample; for instance, such an effort is due to Bruns [2]. Meanwhile, the study of Oda's Oberwolfach question has led to a considerable number of beautiful results on lattice polytopes which were obtained by a variety of authors [1, 6, 7, 8].

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