Brackets, trees and the Borromean rings

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We describe some of the beautiful mathematical structures that arise from the study of the associativity equation. Our journey takes us from combinatorics to abstract algebra, with brief excursions through geometry and topology along the way.

1 The combinatorics of bracketings

In this snapshot, we will have a glimpse at the wonderful complexity that emerges from the familiar associativity equation:

$$(ab)c = a(bc).$$

This equation tells us that we get the same result, whether multiplying a and b first, and multiplying the result with c or multiplying b and c first, and multiplying a second. However, in the following, we will not look at calculations. Rather, we concentrate on the many different ways to use brackets in such equations, and the number of combinations. Let us warm up with the following counting problem considered by Catalan [3] in the 1800s:[2]

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[2] This problem is equivalent to that of counting the number of triangulations of a regular convex polygon proposed by Euler in the 1750s.

Consider a string of $n + 1$ letters, say $a_0a_1a_2 \cdots a_n$. How many distinct ways are there to insert n pairs of correctly matched parentheses so that these define n binary products on this string?

For strings with one to four letters, the different combinations can be easily determined by writing them down as follows. A string with one letter a_0 has a single bracketing without actual brackets. There is also a single bracketing (a_0a_1) for a string with two letters. Here we have only $n = 1$ pair of correctly matched parentheses. For a string with three letters, there are $n = 2$ pairs of correctly matched parentheses, and there are exactly two different ways to insert them: $((a_0a_1)a_2)$ and $(a_0(a_1a_2))$. If we now consider the case of a string with four letters, we see that there are five distinct ways to insert the $n = 3$ pairs of parentheses: $(((a_0a_1)a_2)a_3)$, $((a_0(a_1a_2))a_3)$, $((a_0a_1)(a_2a_3))$, $(a_0((a_1a_2)a_3))$ and $(a_0(a_1(a_2a_3)))$. Now we arrange these five different combinations as follows:

$$\begin{array}{lll}
\textit{group } 1: & (\underline{a_0}((a_1a_2)a_3)) & (\underline{a_0}(a_1(a_2a_3))) \\
\textit{group } 2: & (\underline{(a_0a_1)}(a_2a_3)) & \\
\textit{group } 3: & (\underline{((a_0a_1)a_2)}a_3) & (\underline{(a_0(a_1a_2))}a_3)
\end{array}$$

We see that the first inner pair of parentheses divides the string into a left block (which is underlined), and a right block (which is not underlined). We can use this observation for the general case of the string $a_0a_1a_2 \cdots a_n$. Here, we have the left block $a_0a_1 \cdots a_k$, and the right block $a_{k+1} \cdots a_n$. Any bracketing of a string with $k + 1$ letters can appear within the left block $a_0a_1 \cdots a_k$. Similarly, any bracketing of a string with $n - k$ letters can appear within the right block $a_{k+1} \cdots a_n$. In the example above, we had $k = 0$ for group 1, $k = 1$ for group 2 and $k = 2$ for group 3. The number C_{n+1} of distinct bracketings on our string $a_0a_1a_2 \cdots a_n$ is

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad \textit{for} \quad n \geq 0 \quad \textit{and} \quad \textit{with} \quad C_0 = 1.$$

This is called *Segner's Recurrence Formula.* Regarding the strings with one to four letters, we had the following numbers of distinct bracketings:

$$\begin{aligned}
C_0 &= 1 \quad \text{for} \quad a_0, \\
C_1 &= 1 \quad \text{for} \quad a_0a_1, \\
C_2 &= 2 \quad \text{for} \quad a_0a_1a_2, \\
C_3 &= 5 \quad \text{for} \quad a_0a_1a_2a_3.
\end{aligned}$$

Using these numbers in Segner's Recurrence Formula will give us the number of different ways to insert four pairs of matching parentheses to a string of five letters $a_0a_1a_2a_3a_4$:

$$\begin{aligned} C_4 &= C_0C_3 + C_1C_2 + C_2C_1 + C_3C_0 \\ &= 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 1 + 5 \cdot 1 \\ &= 14. \end{aligned}$$

While we have found a way to count the different combinations of brackets, it would be very tedious to count the combinations for longer strings recursively. Instead, there is a more elegant way to do so. From Segner's Recurrence Formula, one can derive the closed-form expression [3]

$$C_n = \frac{1}{n+1} \cdot \binom{2n}{n}, \qquad n \geq 0,$$

although the derivation is not completely straightforward. The resulting numbers are [4]

$$1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430,\ 4862,\ 16796,\ \ldots$$

They are collectively known as the *Catalan numbers*, and they appear remarkably often throughout mathematics. Stanley's book [15] lists 214 families of mathematical objects that are counted by the Catalan numbers, and new families are discovered on a regular basis. [5] Isn't it remarkable that such a fundamental sequence of numbers arises from simple considerations about bracketings?

2 The Tamari lattices

One of the insights of 20th-century mathematics is the importance of recognising and keeping track of the structures that are present among the objects that one investigates. Such a structure is, for example, the smaller relationship between two numbers. In the case of natural numbers we can always recognise which element is "smaller" and we have a sense of what "smaller" means. We indicate this by the less-than-or-equal relation $\leq$. More generally, we speak of a "partial order". A *partial order on a set X* consists of a relation $x \preceq y$ that satisfies the following axioms:

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[3] In this formula, the expression $\binom{2n}{n}$ is called the binomial coefficient and is read as "2n choose n". If you have a set with $2n$ elements, there are $\binom{2n}{n}$ different ways to choose n elements from this set. The binomial coefficient is used in many areas of mathematics.

[4] See the corresponding entry (A000108) in the Online Encyclopedia of Integer Sequences.

[5] See for example Snapshot 4/2021 [14], where the previous argument is applied to count a family of objects appearing in the theory of quiver representations.

Reflexivity For each element $x \in X$, we have $x \preceq x$.

Antisymmetry If $x \preceq y$ and $y \preceq x$, then $x = y$.

Transitivity If $x \preceq y$ and $y \preceq z$, then $x \preceq z$.

Thus, a partial order tells us when an element of the set is "smaller" than another one with the caveat that two arbitrary elements are not necessarily comparable.[6] In his 1951 doctoral dissertation [17], Tamari observed that the set of bracketings of a string with $n+1$ letters has the structure of a partially ordered set. For this purpose, we declare that the bracketing $(a_0a_1)a_2$[7] is smaller than the bracketing $a_0(a_1a_2)$, and indicate this relation with an arrow:

$$(a_0a_1)a_2 \longrightarrow a_0(a_1a_2).$$

More generally, for a fixed string of letters, we say that one bracketing is smaller than another one if the latter can be obtained from the former by a sequence of rightwards applications of the associativity law. The resulting structures are called *Tamari lattices*.[8] Figure 1 depicts the Tamari lattice for a string with four letters, with an arrow pointing from a smaller element to a larger one obtained by a single rightwards application of the associativity law.

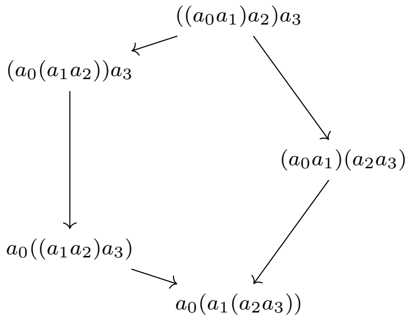


Figure 1: The Tamari lattice for a string with four letters.

The Tamari lattice for a string with five letters has a three-dimensional character and is depicted in Figure 2.

[6] This is quite different from the usual order $\leq$ on the natural numbers, where given two natural numbers a and b we have that $a \leq b$ or $b \leq a$.

[7] From this point on, the outermost pairs of matching parentheses are irrelevant to what follows. They have been omitted to increase readability.

[8] A lattice is a special kind of partially ordered set.

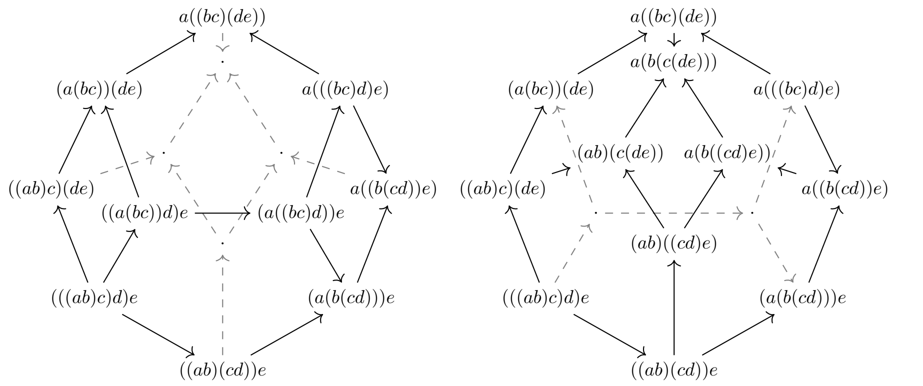
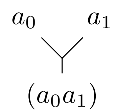


Figure 2: The Tamari lattice for a string with five letters.

We invite the interested reader to peruse the volume [13], published to commemorate the centennial anniversary of Tamari's birth, which contains rather accessible surveys and interesting historical remarks on a wide range of mathematics related to the Tamari lattices (some of them discussed in this snapshot). For us, the illustrations of the Tamari lattices in Figures 1 and 2 suggest a relationship between bracketings and *polytopes* – higher-dimensional analogues of polygons and polyhedra – that we outline in the next section.

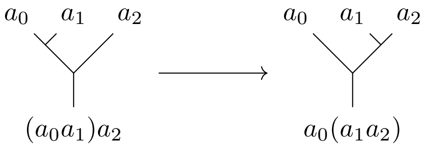
3 From associativity to mathematical dendrology

Before continuing our discussion, it is convenient to introduce a way to visualise different bracketings. Namely, each bracketing of a string with $n+1$ letters corresponds to what mathematicians call a "planar binary rooted tree with $n+1$ leaves." Rather than giving a precise definition of this mouthful, let us look at some examples. The bracketing $(a_0 a_1)$ corresponds to the following tree:



(1)

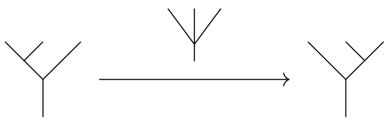
This tree has two leaves at the top and one root pointing to the bottom – it is called "planar" because it comes with a specific way of being drawn in the plane. The two bracketings $(a_0a_1)a_2$ and $a_0(a_1a_2)$ correspond to the trees



(2)

which now have three leaves and one root. As before, we have indicated the rightwards application of the associativity law with an arrow. All of these are binary trees since there are exactly two branches emanating upwards from every node.[9] Thus, instead of bracketings, we may consider the Tamari lattices as partial orders on planar binary rooted trees with a fixed number of leaves.

The interpretation of the Tamari lattices in terms of trees reveals further structure that is not immediately apparent from looking at bracketings. Consider the two planar binary rooted trees with three leaves, and notice that they both have two junctions, by which we mean a branching point in the tree. The root of the tree connects to the first junction, from which two "big" branches grow. Then comes what we call an "internal edge" that connects the first junction to the second one, from which two "little" branches grow. Each of the trees has a unique internal edge. If we contract this edge to one point, we obtain a new planar rooted tree that is no longer binary since three branches/leaves emanate from the unique node. We record the outcome of this procedure pictorially as follows:[10]



The Tamari lattice for the planar binary rooted trees with four leaves is depicted in Figure 3. Notice that each of the binary trees that is placed on the vertices of the pentagon now has *two* internal edges that we can contract separately into a point. Something conspicuous happens: The new non-binary trees that label the edges of the pentagon each have a single internal edge. If, in any of these trees, we contract this edge to a point we obtain the last planar rooted tree with four leaves. We have included this tree in the Tamari lattice as a further decoration, this time as a label of the interior of the pentagon.

[9] Planar binary trees are perhaps better known as genealogical trees, although they are typically drawn upside down in that context.

[10] Now at the latest, the relation $\preceq$ indicated by the arrow looks more like a transition. This applies to the description of the figures in this section as a process.

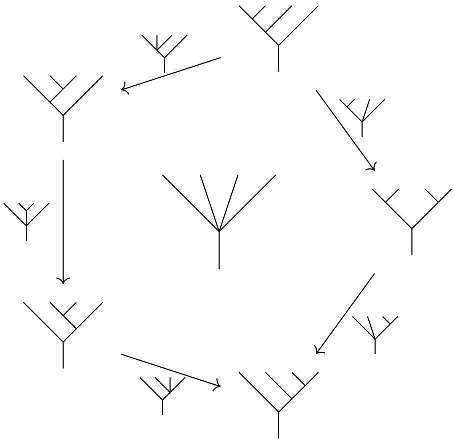


Figure 3: The decorated Tamari lattice for trees with four leaves.

In general, every planar rooted tree can be obtained from the binary ones by contracting one internal edge at a time. Remarkably, the planar rooted trees with $n+1$ leaves label the cells of an $(n-1)$-dimensional polytope that is called an *associahedron*. This name is a reference to the associativity equation from the beginning of our journey. It marks the connection between the polytope and its origin.[11]

The advanced reader might appreciate the close relationship between associativity and planar rooted trees. It can be used, among other things, to establish combinatorial formulas for the inverse with respect to composition of formal power series [11]. As a curious example, the formal power series

$$h(t) = \sum_{n\geq 0} (-1)^{n+1} C_n t^{3n+1},$$

whose coefficients are the Catalan numbers up to a sign, is its own inverse: $h(h(t)) = t$.

Ceballos and Ziegler [4] remind us that associahedra have been described by Haiman [6] as "mythical polytope[s]" and, as we shall see below, there is more to the relationship between associahedra and the associativity equation than meets the eye ...

[11] The planar rooted trees with $n+1$ leaves and k internal edges label the $(n-1-k)$-dimensional cells of the corresponding associahedron. It is a nice exercise to label the three-dimensional associahedron in Figure 2 with the planar rooted trees with five leaves.

4 Stasheff's A_∞-algebras

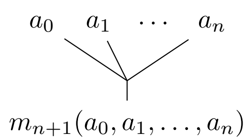
Although associahedra were already present in Tamari's thesis, they were rediscovered in the 1960s by Stasheff – also in his doctoral dissertation [16]. Stasheff's motivation came neither from combinatorics nor from geometry, but from the field of algebraic topology. In a nutshell, algebraic topology studies the qualitative properties of spaces, such as the number of holes in a surface, using tools from abstract algebra. Through his investigations, Stasheff discovered a new class of algebraic structures that he called A_∞-algebras. The precise definition is rather technical, so we only sketch some of the main ideas behind it.

An ordinary algebra is a collection of elements that can be added, subtracted and multiplied, just like real numbers. However we cannot necessarily divide within an algebra and the multiplication does not necessarily satisfy the commutativity law. For example, the polynomials with a fixed number of variables and coefficients in the real numbers form an algebra, and so do the $n \times n$ matrices with real entries. What is important for our discussion is that the multiplication in an algebra is a binary operation: it takes as input two elements a and b and outputs their product ab. Thus, we may visualise the multiplication operation in an algebra as a process described by the binary tree in (1). More generally, iterated applications, keeping track of the corresponding bracketing, of the multiplication operation in an algebra can be encoded with a planar binary rooted tree! For example, the two ways to combine the multiplication operations in order to multiply three elements correspond to the two trees in (2).

What about the planar rooted trees that are non-binary? Stasheff discovered that some structures that arise in algebraic topology have not only binary operations, such as multiplication, but a possibly infinite system of "higher operations" with $n + 1$ inputs and a single output:

$$(a_0, a_1, \ldots, a_n) \longmapsto m_{n+1}(a_0, a_1, \ldots, a_n), \qquad n \geq 0.$$

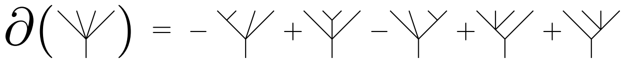
Crucially, the operation m_{n+1} is not obtained by iterated applications of the binary operation $m_2(a_0, a_1) = a_0 \cdot a_1$ but is rather a new operation that combines $n + 1$ elements "all at once". With this in mind, it is natural to visualise the operation m_{n+1} as a process described by the following tree:



Arbitrary planar rooted trees are then obtained by combining the various higher operations. There are an infinite number of these higher operations m_{n+1}, since there exists one for each possible number of inputs. Stasheff also discovered that these operations satisfy an infinite system of equations, called the A_∞-*equations*, that serve as a replacement of the associativity equation:

$$\sum_{r+s+t=n+1} \pm m_{r+1+t}(a_0, a_1, \ldots, a_{r-1}, m_s(a_r, \ldots, a_{r+s-1}), a_{r+s}, \ldots, a_n) = 0.$$

We can think of these equations as a variant of the associativity equation from the beginning of this snapshot: $(ab)c - a(bc) = 0$. We used a slightly different representation there. Here we have $n+1$ inputs instead of three. Still, the formula expresses the same idea: We apply all possible combinations of two operations to these $n+1$ elements, and adding or subtracting over all combinations gives 0. The reader is not expected to parse the A_∞-equations easily (mathematicians struggle to do this the first time they see them as well). We merely wish to highlight the kind of complexity that arises in this context. However, this complexity is not arbitrary: As it turns out, the A_∞-equations are governed by the combinatorial structure of the Stasheff–Tamari associahedra, which we discussed in the previous section. For example, rewritten in terms of a certain "boundary operator" $x \mapsto \partial(x)$, the A_∞-equation corresponding to the value $n = 3$ involves precisely the interior and the boundary of the two-dimensional associahedron:[12]



In the above depiction of the A_∞-equation, we have replaced combinations of higher operations by the corresponding trees.

Summarising, an A_∞-algebra consists, roughly speaking, of a collection of elements that can be added, subtracted and that are equipped with a system of operations that satisfy the A_∞-equations. One of the many marvels of A_∞-algebras is that they appear not only in algebraic topology but also in fields as diverse as algebraic geometry, representation theory and symplectic geometry. For example, they are crucial to Kontsevich's famous Homological Mirror Symmetry Conjecture that foresees a far-reaching bridge between algebraic and symplectic geometry [10].

Having scratched only the surface of A_∞-algebras, we leave the abstract realm, to take a closer look at a more concrete example in the next section.

[12] The signs that appear are related to the arrows in the Tamari lattice. Do you see how? Answer: We could imagine to walk along the edge of the associahedron from Figure 3. If we walk counterclockwise, we follow the arrows on the left side all the way down. To complete a full circle, we follow the arrows in their negative direction, hence the minus sign in the formula. We continue walking counterclockwise upwards and end up where we started. We can imagine the boundary operator as the desribed walk.

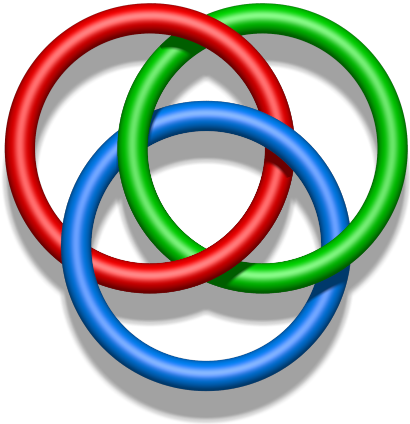


Figure 4: The Borromean rings.

5 The Borromean rings

Although the definition of an A_∞-algebra is quite technical and somewhat obscure, the higher operations sometimes have interesting interpretations. Here is one such example.

The Borromean rings, depicted in Figure 4, are a configuration of three interlinked circles in three-dimensional space.[13] The distinctive feature of this configuration is the way the three rings are linked together: If we remove any one circle the remaining two are no longer linked and can be moved away from each other.[14] Using sophisticated machinery from algebraic topology, one constructs an A_∞-algebra from the Borromean rings; this A_∞-algebra has three distinguished elements α, β and γ (corresponding roughly to the three circles in the configuration) whose multiplication table looks as follows:

$$\alpha\beta = \beta\alpha = 0, \quad \alpha\gamma = \gamma\alpha = 0 \quad \text{and} \quad \beta\gamma = \gamma\beta = 0.$$

These equations reflect the fact that any two circles in the Borromean rings are unlinked. On the other hand, Massey [12] discovered (in a slightly different language) that there is a ternary operation that is not always zero

$$m_3(\alpha, \beta, \gamma) \neq 0.$$

[13] They are named after the House of Borromeo, who included them in their coat of arms.

[14] An elementary proof of the fact that the Borromean rings are indeed linked can be found in the book [1] by Aigner and Ziegler, a book that we wholeheartedly recommend to the reader.

The existence of this operation witnesses the fact that the circles in the Borromean rings are triply linked!

6 A_∞-algebras in contemporary mathematical research

Even though A_∞-algebras were invented about sixty years ago, many of their properties remain mysterious and their careful study has led to unexpected applications. For example, in their work in three-dimensional algebraic geometry [5], Donovan and Wemyss formulated a deep conjecture that, thanks to the work of several mathematicians, relates a beautiful family of geometric objects called compound Du Val singularities[15] to a particular class of A_∞-algebras. The recent solution [7] to the conjecture[16] involves a delicate analysis of the qualitative properties of these A_∞-algebras – an approach that was not at all obvious from the original formulation of the conjecture!

[15] See also Snapshot 7/2014 [2], where the (two-dimensional) Du Val singularities make an appearance.
[16] See also [8, 9].

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