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Trisections of four-dimensional spaces

Sarah Blackwell[1]

This snapshot introduces the theory of trisections of smooth 4-manifolds, an area of exploration in low-dimensional topology aiming to make four-dimensional spaces more understandable. Along the way, we discuss the concepts of topology, dimension, manifolds, and more!

1 The mysterious fourth dimension

Four-dimensional spaces are notoriously hard to study. One might think that the higher the dimension, the harder it is to study, but strangely enough, this is not the case.[2] Above dimension 4, loosely speaking, there is enough "room" to study these spaces by moving around lower-dimensional spaces within them.[3] Dimensions 1, 2, and 3 have the benefit of being easy to visualize, and much work has been done classically in these dimensions. However, dimension 4 remains mysterious, acting as a sort of "phase change" between lower and higher dimensions: it is both too big and not big enough.

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[2] From certain perspectives. The author discloses that she does not personally study dimensions higher than four and is thus highly biased.

[3] This is a vague reference to a theorem called the *h-cobordism theorem*.

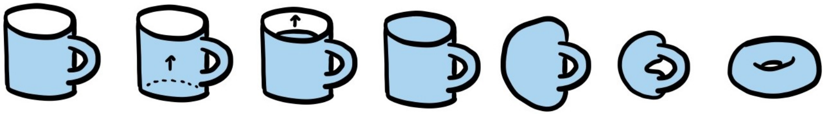

Figure 1: A coffee mug morphing continuously into a doughnut. A very nice animation of this can be found at the top of the Wikipedia page for *homeomorphism* [30].

The study of the fourth dimension is of inherent interest to pure mathematicians, but there are applications of this study outside of theoretical mathematics as well. For instance, physicists use four-dimensional space to model space-time, with three dimensions representing "space" and one dimension representing "time". So, research that helps demystify the fourth dimension is of broad interest to the scientific community.

2 Setting the stage

One subfield of pure mathematics that is especially invested in the study of the fourth dimension is *low-dimensional topology*, where dimensions 4 and less count as "low". Topology is the study of continuous deformations of geometric objects; a long-running joke says that topologists cannot tell the difference between a doughnut and a coffee mug, because (if they were made out of playdough) one could be molded continuously (without breaking or gluing) into the other, as in Figure 1.

2.1 The main characters: manifolds

The main characters in the field of low-dimensional topology, that is, the objects primarily being studied, are called "manifolds". Of all of the kinds of geometric objects that might exist, manifolds are particularly nice, as they have certain properties that make them appealing and accessible for research. They are also natural in some sense, showing up in other disciplines, such as physics, and "real world" situations, as we will see shortly.

Readers may recall the term *Euclidean space* from a high school or college mathematics course. One-dimensional Euclidean space is a line extending infinitely in both directions. If one chooses an origin, one can associate a real number to each point on the line. That is why one-dimensional Euclidean space is also called the *real number line*. Two-dimensional Euclidean space may be particularly familiar; this is the plane on which equations are graphed using Cartesian coordinates. In other words, the plane is obtained by taking two real

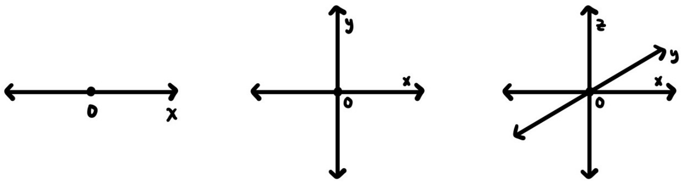

Figure 2: From left to right: models for one-, two-, and three-dimensional Euclidean space. Each is built from the previous one by adding a new axis.

number lines and intersecting them at their origin. The position of each point in the plane is then determined by two real numbers, read from the two axes. Three-dimensional Euclidean space takes the Cartesian plane and adds a third axis; see Figure 2. Beyond three dimensions, we lose the ability to visualize this space, but the definition can be extended similarly. Four-dimensional Euclidean space is thus spanned by four axes, and one needs four real numbers to locate a point in four-dimensional space.

Euclidean space is, in some sense, extremely manageable. For instance, this is the setting in which we learn calculus. *Manifolds* are, in some sense, the next best thing; these are objects which *locally* look like Euclidean space, even though they might *globally* look like something else. This means that around each point on a manifold, there is a small region that can be continuously deformed (like the coffee mug in Figure 1) so that it looks like a region in Euclidean space. For example, the surface of the Earth is a two-dimensional[4] manifold called a *sphere*. The reason is that locally, the Earth looks like a flat plane; when we stand on the Earth's surface and look in all directions, it looks flat to us, and indeed humanity thought the Earth was flat for a large part of our existence. However, we now know that the Earth is indeed curved, and if we were to view the Earth from outer space, we could see this curvature; see Figure 3.

Two important types of manifolds, which often serve as building blocks for constructing more complicated manifolds, are *genus g surfaces* and *handlebodies*. Surfaces are two-dimensional manifolds described as follows. The genus 0 surface is the sphere, and the genus 1 surface is the torus, which resembles an inner tube.

[4] Some readers may object to "two-dimensional", as opposed to "three-dimensional". This is because outside of research mathematics, it is a common practice to refer to spheres (and certain other shapes) as three-dimensional because they are being thought of as sitting inside three-dimensional space. However, inherently the sphere is a two-dimensional object because it locally looks like a plane, which is two-dimensional Euclidean space. It should also be noted that while the *surface* of the Earth is a two-dimensional space, the Earth itself (including all the stuff below the surface) is three-dimensional.

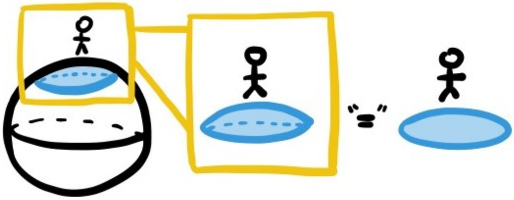

Figure 3: An observer (not to scale) standing on the surface of the Earth, which is a sphere. A small region around the observer is a disk, and although in reality it is curved like a contact lens, it can be flattened out continuously to look like a region in two-dimensional Euclidean space.

For $g > 0$, genus g surfaces are obtained by sticking g copies of the torus together; see Figure 4 (top row). The non-negative integer g then counts the number of "holes" in the surface. A handlebody is a three-dimensional manifold obtained by "filling in" a two-dimensional genus g surface; see Figure 4 (bottom row).

2.2 Two flavors: smooth versus topological

There are some other technicalities involved in the precise definition of a manifold, but for this article, it is mostly enough to think of manifolds as objects that locally look like Euclidean space. However, when thinking about four-dimensional manifolds, or 4-*manifolds*, there is one other technical consideration to take into account. There are different flavors of manifolds, and two of the most important types to consider in this dimension are called *smooth* and *topological*. In dimensions 1, 2, and 3, there is no difference between a smooth and a topological manifold,[5] but in dimension 4, there can be quite a large difference, and in fact, this is one of the reasons why dimension 4 is so mysterious!

The distinction between the two – in a very broad sense – is that we can do calculus on smooth manifolds, but not on topological manifolds. All smooth manifolds are also topological manifolds, but not all topological manifolds are smooth manifolds; topological manifolds allow "corners" which mess up our ability to do calculus. Readers might remember a classic example from their calculus class: the absolute value function $y = |x|$ is not differentiable at the corner $(0, 0)$. Therefore, the graph of this function is not a smooth 1-manifold, but it is still a topological manifold, as the corner can be continuously deformed so that it looks like one-dimensional Euclidean space again.

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[5] More explicitly, in these dimensions, two manifolds cannot be topologically the same but smoothly different. However, this phenomenon *can* happen in dimension 4!

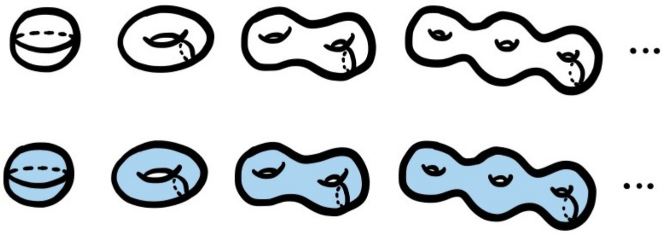

Figure 4: Genus g surfaces (top row) and handlebodies (bottom row). The surfaces in the top row do not have "stuff" inside them; for instance, think of the second picture as an inner tube. The handlebodies in the bottom row are obtained by "filling in" the surfaces; for instance, think of the second picture as a solid doughnut. Left to right, the genera of the surfaces shown are 0, 1, 2, and 3.

Both topological and smooth manifolds are of interest to low-dimensional topologists, but in this article we focus on smooth manifolds, as this is the setting in which we can enact our chosen methodology for studying the fourth dimension: trisections![6][7]

2.3 The methodology: trisections

Often the techniques used in the study of four-dimensional spaces are inspired by techniques that have worked in three dimensions. In the study of three-dimensional manifolds, or 3-*manifolds*, it is often useful to consider decompositions called *Heegaard splittings*. Every 3-manifold admits a splitting into two "nice" pieces, and this decomposition allows the 3-manifold to be described by diagrams consisting of curves on a surface. A trisection is the four-dimensional analogue of this idea; a trisection decomposes a 4-manifold into three "nice" pieces, with the complexity of the manifold relegated to the way these pieces are glued together, and allows a similar diagram.

The theory of trisections, first introduced in 2012 by Gay and Kirby [12], is a relatively new field which has inspired a great amount of work in the

[6] Additionally, what is widely considered to be the most important open problem in the field of low-dimensional topology deals with smooth 4-manifolds. Grigori Perelman famously solved the *Poincaré conjecture* – and subsequently declined a Fields Medal and Millennium Prize for this breakthrough – which was a conjecture about 3-manifolds. The conjecture had previously been posed and partially answered for other dimensions and categories of manifolds, but it is still not known whether it is true for smooth 4-manifolds.

[7] For further reading on the distinction between smooth and topological, see for instance the general audience articles [13, 14] from *Quanta Magazine*.

years since its introduction. These decompositions are useful in the study of 4-manifolds as they provide a way to turn smooth information (the 4-manifold) into combinatorial data (the diagram). An especially intriguing aspect of trisections is their apparent ability to encode geometric information efficiently, as evidenced by much recent work developing the connections between trisections and other subfields of pure mathematics such as *complex geometry* and *symplectic geometry* [10, 3, 9, 19, 20, 16, 18, 4].

3 Getting into the details

Originally [12], trisections were defined for smooth 4-manifolds which also satisfied the following properties: "closed", "connected", and "orientable". Subsequently, the definition has been extended to include 4-manifolds that do not have some of these properties [6, 7, 8, 27], as well as various other settings such as dimensions other than 4. We will expand briefly on this later.

3.1 The definition

For now, we focus on the original setting – smooth 4-manifolds which are also "closed", "connected", and "orientable". Roughly speaking, *closed* means that the manifold has no boundary and can be contained inside of a ball (that is, it does not extend forever),[8] *connected* means that it consists of only one piece, and *orientable*[9] means that it has an "inside" and an "outside".[10] However, requiring these properties is not very restrictive; there are many closed, connected, orientable, smooth 4-manifolds.

As described previously, a trisection is a decomposition of a 4-manifold into three "nice" pieces. This is often represented with the schematic in Figure 5. The disk altogether represents the 4-manifold which we call X,[11] and the sectors represent the three pieces, which are also four-dimensional and which we call X_1, X_2, and X_3. These pieces are required to be *four-dimensional handlebodies*.[12]

[8] For example, the disk is not closed while the sphere is closed, because an observer walking on the surface of the Earth will never reach a boundary (as would be the case with walking on a disk). The Earth itself as a 3-manifold is not closed: it is bounded by the sphere.

[9] Two well-known surfaces which are non-orientable are the *Möbius strip*, which has a boundary, and the *Klein bottle*, which does not have a boundary.

[10] All the surfaces shown in Figure 4 are closed, connected and orientable, while the handlebodies are connected and orientable, but not closed, as they have boundaries.

[11] This schematic is a bit misleading, because the manifold we are considering does not have a boundary, while the disk does have a boundary. A better schematic would be a trisected sphere, but this is harder to draw and digest.

[12] Technically speaking, these are called four-dimensional *one*-handlebodies, as there are different types of handlebodies in this dimension. But we will not discuss this in more depth, so we remove the "one" for convenience.

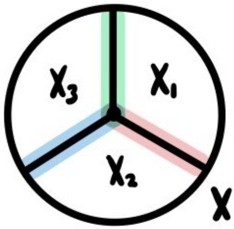

Figure 5: A schematic representing a 4-manifold X which has been split into three pieces X_1, X_2, and X_3.

We will not define them here, but one can think of them as four-dimensional analogues of the three-dimensional handlebodies. The three spokes in the disk, colored red, blue, and green, represent the intersections of the four-dimensional pieces: $X_1 \cap X_2$ (red), $X_2 \cap X_3$ (blue), and $X_3 \cap X_1$ (green). These intersections are three-dimensional pieces which are required to be handlebodies. The point in the center of the disk where the spokes meet is the triple intersection of the four-dimensional pieces: $X_1 \cap X_2 \cap X_3$. This must be a genus g surface, which is referred to as the *central surface*. The non-negative integer g is called the *trisection genus*.

3.2 Some musings

This decomposition of a 4-manifold into four-dimensional handlebodies is particularly nice for two reasons: (1) it is well-understood what "four-dimensional handlebodies" look like (although we will not go into detail about this here), and (2) due to a result by Laudenbach and Poénaru [22], the two- and three-dimensional pieces determine the four-dimensional pieces.[13]

When Gay and Kirby defined trisections, they proved that every 4-manifold (with the extra properties noted above) admits such a decomposition, and furthermore, this decomposition is unique up to a "stabilization" operation. This means that although a given 4-manifold does not admit one unique trisection on the nose, the different trisections it admits can all be related by stabilization. We will not define it here, but this operation increases the trisection genus in a natural way, and mirrors a similar operation in the setting of Heegaard splittings.

[13] A common way of saying this is that given the two- and three-dimensional stuff, the four-dimensional handlebodies can be "glued in" uniquely.

Figure 6: Some examples of genus 0, 1, and 2 trisection diagrams. The sphere without curves is a very special case; in general, a genus g surface will require g red, blue, and green curves each in a diagram.

3.3 Diagrams

The fact that trisections are completely determined by their two- and three-dimensional parts allows us to represent a trisected 4-manifold by a diagram consisting of curves on the central surface. This is particularly useful, as the fourth dimension is naturally very difficult to visualize, and these diagrams allow us to see how the 4-manifold is built in dimensions that we understand. Some examples of trisection diagrams are given in Figure 6.

Roughly speaking, the curves on the central surface show how to "glue in" the three-dimensional handlebodies;[14] in this way, the colors of the curves in Figure 6 (red, blue, and green) correspond to the colors of the handlebodies in Figure 5 (red, blue, and green). Once we know how the handlebodies are attached to the central surface, we have determined the entire trisected 4-manifold, by Laudenbach and Poénaru [22].

Given a value g of the trisection genus, not every 4-manifold is guaranteed to admit a trisection of this genus, although it *is* guaranteed that the 4-manifold admits a trisection of *some* genus. The stabilization operation increases the genus by one, which means that while the genus can always be increased, for each 4-manifold there is a *minimal* genus for which the 4-manifold admits a trisection of that genus. The diagrams shown in Figure 6 are the only genus 0, 1, and 2 trisection diagrams.[15][16] This is not very difficult to prove for the genus 0 and 1 cases, but solving the genus 2 case required a considerable amount of effort [25]. Such a classification for genus 3 and higher is unknown, although due to work of Meier [23], in stark contrast with the situation for genus 2 and lower, it is known that infinitely many smooth 4-manifolds admit genus 3 trisections!

[14] We will not explain how that works here, but it is interesting that the three-dimensional handlebodies can be determined with just one-dimensional curves!

[15] This is up to certain moves on the diagrams which preserve the trisection of the 4-manifold.

[16] By an operation called "connected sum", one can build higher genus diagrams from multiple smaller genus diagrams. Technically, the diagrams in Figure 6 are the only *irreducible* diagrams, meaning they are not the connected sum of smaller genus diagrams.

4 Odds and ends

4.1 What other kinds of things can be trisected?

The notion of trisections has been extended to many other settings; we will briefly mention a few here. The study of so-called “knotted surfaces” in 4-manifolds is akin to the study of knots in three-dimensional spaces. A knotted surface sitting inside of a 4-manifold can always be put into a position such that it inherits a trisection of its own [24, 26, 15]. Another extension is the algebraic setting of “group trisections”, in which an algebraic object called a “fundamental group”, corresponding to a 4-manifold or knotted surface, is trisected [1, 5]. Finally, extensions to various higher-dimensional settings have also been considered [21, 28, 2].

4.2 What is so special about the number three?

In short, nothing. Islambouli and Naylor generalized the notion of trisections to “multisections” [17], in which the number of pieces the manifold is split into increases. One benefit of using more pieces is that generally, as the number of pieces increases, the multisection genus decreases, which might be desirable depending on the context. However, going smaller than three – trying to “bisect” a 4-manifold – is where one runs into problems. Generally, at least three pieces are required in order for this decomposition to work, although the notion of a “bisection” has been explored in various special settings [17, 18].

4.3 So you want to be a topologist?

For those interested in further reading, there are many great places to look. For an excellent introduction to the world of topology with a low barrier to entry, check out “The Shape of Space” by Weeks [31]. An advanced undergraduate student or graduate student interested in the fourth dimension may wish to check out “The Wild World of 4-Manifolds” by Scorpan [29]. Finally, for the graduate student or mathematics researcher looking for an introduction to trisections, the paper “From Heegaard splittings to trisections; porting 3-dimensional ideas to dimension 4” by Gay may be of interest [11].

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