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The five Platonic solids and their connection to root systems

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Platonic solids have fascinated humans for thousands of years. In ancient times, they were associated with the elements fire, air, water, earth, and aether. These solids are completely symmetrical three-dimensional polyhedra. In this snapshot, it is first explained that there can only be five such polyhedra in the three-dimensional space. For this purpose, so-called Schläfli symbols and Coxeter graphs are introduced. More precisely, the (linear) Coxeter graphs correspond to the (linear) Schläfli symbols that, in turn, correspond exactly to the regular convex polyhedra. Through this one-to-one relationship, it is possible to classify the regular convex polytopes in any dimension by exploiting the classification of Coxeter graphs.

1 Preliminaries

First of all, to get notation and language clear, we introduce basic definitions on polytopes. In this snapshot, we focus on two- or three-dimensional (convex) polytopes. This section follows the book [3] of H. S. M. Coxeter.

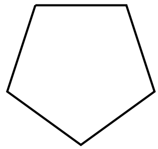


Figure 1: A convex pentagon, denoted by $\{5\}$.

1.1 Polygons

In the following, a *two-dimensional polytope*, also called *polygon* or *p-gon*, is a shape obtained by starting with p points in the plane and going from one point to the next through straight line segments – eventually ending up at the first point again while the line segments never cross. The points and segments are called *vertices* and *sides*. A polygon separates the plane into two regions: the one inside the polygon and the one outside of it. The region inside the polygon is called the *interior* of the polygon. We will focus on the case of *convex* polygons, that is, when any line connecting two points on any two sides of the polygons lies in its interior. A polygon is said to be *equilateral* if all sides have the same length, *equiangular* if all angles have the same magnitude, and *regular* if it is both equilateral and equiangular. We denote a convex regular p-gon with the symbol $\{p\}$, see Figure 1 for an example. This symbol is called the *Schläfli symbol* of the polygon.

1.2 Polyhedra and Platonic solids

A *polyhedron* or *three-dimensional polytope* may be defined as a finite, connected set of non-crossing two-dimensional polygons, such that every side of each polygon belongs to just one other polygon, providing that the polygons surrounding each vertex form a single circuit (see Figure 2). The polygons are called *faces*, and their sides *edges*. We define the *interior* of the polyhedron as the region inside of it and we say that the polyhedron is *convex* if any line between two points on any two faces lies in its interior. A convex polyhedron is said to be *regular* if its faces are regular two-dimensional polygons of the same type, while all vertices are surrounded by the same number of faces. We call a regular convex polyhedron a *(three-dimensional) Platonic solid*. We define the *Schläfli symbol* of the Platonic solid as the pair $\{p, q\}$ if its faces are p-gons and q of them surround each vertex.

There is also a notion of n-dimensional polytopes and of a generalized Schläfli symbol for the n-dimensional Platonic solids (which are convex regular n-dimensional polytopes). For more details, see [2, Section 20].

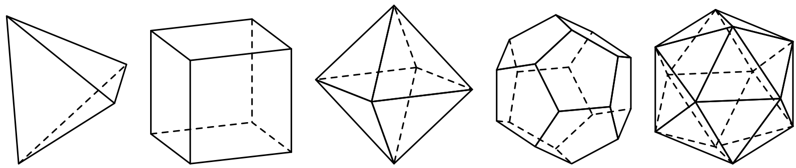


Figure 2: The five three-dimensional Platonic solids.

2 The five Platonic solids in three-dimensional space

In this section, we list the three-dimensional Platonic solids. The main idea of the classification is due to the ancient Greek mathematician Euclid and can be found, for example, in [2] and [3]. The proof needs only fundamental geometry and is easy to understand. The reader is encouraged to read the proof in [2] or [3] to complement this snapshot.

Proposition 2.1. *There are only five Platonic solids in dimension* 3*. Concretely we have:*

$\{3,3\} =$ tetrahedron	$\{5,3\} =$ dodecahedron
$\{4,3\} =$ hexahedron *or* cube	$\{3,5\} =$ icosahedron
$\{3,4\} =$ octahedron	

All Schläfli symbols in the list above give rise to one of the polyhedra in Figure 2.

One can not be sure that, given a Schläfli symbol, the associated polyhedron is one in the sense of the definition in Section 1.2. In fact, there is no guarantee that in the end, after "gluing" faces, the last faces match and can be "glued" together at common edges. However, we can verify that these polyhedra exist by giving a concrete description in coordinates as is done in [4, Section 2.10 and 2.13].

Note that there exists a certain symmetry in the Schläfli symbols in Proposition 2.1: With $\{4,3\}$, we also have $\{3,4\}$, and with $\{5,3\}$, we also have $\{3,5\}$. This is not surprising as for a given convex polyhedron, there is a *dual polyhedron* which can be obtained by connecting the mid-points of the faces of the given polyhedron by an edge whenever the faces are adjacent. This gives rise to a new polyhedron inside the other. In other words, the vertices of the dual polyhedron are the mid-points of the faces of the given one. If a convex polyhedron is regular, so is its dual. Additionally, if $\{p,q\}$ is the Schläfli symbol of a convex regular polyhedron, the dual polyhedron has Schläfli symbol $\{q,p\}$. For example, the tetrahedron is self-dual, the cube and the octahedron are dual

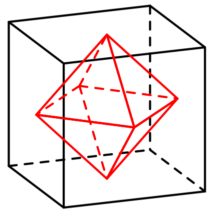


Figure 3: The dual relationship of Platonic solids using the example of the cube and the octahedron.

to one another, and so are the icosahedron and the dodecahedron, see Figure 3 for an example.

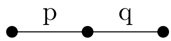
The following example demonstrates another interpretation of the last entry in the Schläfli symbol as a rotation order. This example is crucial for the connection between Platonic solids and Coxeter graphs, as explained later.

Example 2.2. Consider the cube $\{4, 3\}$. The first entry in the Schläfli symbol is the number of vertices of a face, which is a regular 4-gon, that is, a square. Fix now an arbitrary vertex S, draw a line L_S through this vertex and the center of the cube, and rotate the cube about L_S such that, after each rotation, the cube is in the same position but with a different front face. In other words, after each rotation d_S about L_S, the cube looks the same but the faces are exchanged. The rotation d_S is a rotation of angle $120°$ or $\frac{2\pi}{3}$. The second entry of the Schläfli symbol is the *order of* d_S, which is the lowest natural number describing how often we have to rotate the cube such that each face is again at its starting point. Indeed, we have to rotate the cube three times about L_S until the front face becomes again the front, thus $\mathrm{ord}(d_S) = 3$. Because of the regularity of the cube, this number does not depend on the choice of the vertex S.

It is left to the reader to check that also for all other Platonic solids listed in Proposition 2.1, the second entry of the Schläfli symbol is such a rotation number. Note that the first entry of the Schläfli symbol can also be interpreted as rotation order, since it is the second entry of the Schläfli symbol of the dual polyhedron.

In the following definition, we associate a graph to a given Schläfli symbol. A *graph* is a set of points, called *vertices*, connected by lines, called *edges*. The advantage of this association will become clear later.

Definition 2.3. Consider a graph whose vertices are three points and connect the first to the second and the second to the third by a straight line. This gives rise to a graph with three vertices and two edges. Labelling the edges by the entries of the Schläfli symbol gives a correspondence between such a graph and the given symbol. The Schläfli symbols $\{p, q\}$ and $\{q, p\}$ share the same graph, read from the left or right side, that is,

$\{p, q\}$ and $\{q, p\}$ correspond to 

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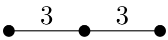
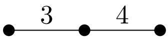
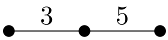
Example 2.4. Since Schläfli symbols are associated to Platonic solids, there is also a correspondence between Platonic solids and the graphs corresponding to the Schläfli symbols:

tetrahedron 

called A_3

cube and octahedron 

called B_3 or C_3

icosa- and dodecahedron 

called H_3

The nomenclature A_3, B_3, C_3, and H_3 is not arbitrary but comes from the theory of root systems. More specifically, we associate to each graph a set of points in the three-dimensional space, called *root system*, which yields the polyhedron we started with by just linking the points by edges.

3 Root systems and their Coxeter graphs

In this section, we introduce the root systems mentioned above. For that, we follow the book [4] of J. E. Humphreys and refer to it for more details and proofs. Root systems are very important objects in many mathematical areas, for example in algebra, geometry, and coding theory. They are highly symmetric and thus it is no surprise that they occur in the context of Platonic solids. We define root systems in complete generality. In particular, we define them in the n-dimensional space $\mathbb{R}^n$, which is the space of points or vectors v with n coordinates, that is $v = (v_1, \dots, v_n)$.

3.1 Reflections and rotations in n-dimensional space

For a vector α of $\mathbb{R}^n$ whose coordinates are all different from zero, we write $\alpha \in \mathbb{R}^n \setminus \{0\}$. Such a vector defines a line L_α through the origin and α and also a *hyperplane* H_α, that is, an $(n-1)$-dimensional subspace, which goes through the origin and is perpendicular to L_α.

A reflection with respect to H_α is defined to be a linear map $s_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ with the properties:

1. $s_\alpha(n) = -n$ for all $n \in L_\alpha$,
2. $s_\alpha(h) = h$ for all $h \in H_\alpha$, and (1)
3. $s_\alpha^2(v) := s_\alpha(s_\alpha(v)) = v$ for all $v \in \mathbb{R}^n$.

For $n = 2$, this describes exactly the usual reflection in a line, given by H_α.

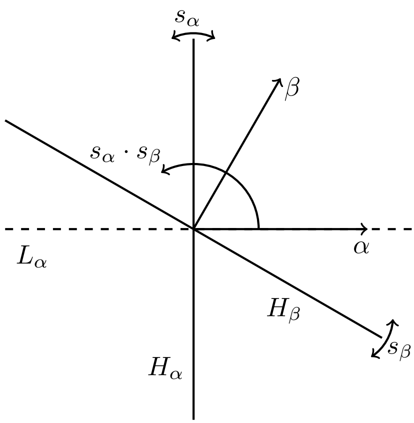


Figure 4: Two reflections s_α and s_β and the corresponding rotation $s_\alpha \cdot s_\beta$ for $n = 2$.

Consider two different vectors $\alpha, \beta \in \mathbb{R}^n \setminus \{0\}$ with different hyperplanes H_α, H_β and their reflections s_α, s_β. The composition of the two maps s_α, s_β is a new map $s_\alpha \cdot s_\beta : \mathbb{R}^n \to \mathbb{R}^n$ given by $s_\alpha \cdot s_\beta(v) := s_\beta(s_\alpha(v))$ and may be interpreted as a *rotation*, see Figure 4. The *angle of rotation* depends on the angle between H_α and H_β. The *order of the rotation* $s_\alpha \cdot s_\beta$ is defined as the lowest natural number k such that $(s_\alpha \cdot s_\beta)^k(v) = v$ for all $v \in \mathbb{R}^n$.

3.2 Root systems

We consider now several vectors, here called roots, and their reflections at once.

Definition 3.1. A *root system* Φ is a finite set of vectors $\alpha \in \mathbb{R}^n \setminus \{0\}$, called *roots*, satisfying the following two properties:

(R1) For each root $\alpha \in \Phi$ and L_α the line in $\mathbb{R}^n$ through the origin and α, we have $\Phi \cap L_\alpha = \{\pm\alpha\}$. That means that there are no other roots in the direction of α than α itself and $-\alpha$.

(R2) For each root $\alpha \in \Phi$, we have $s_\alpha(\beta) \in \Phi$ for all $\beta \in \Phi$. That means that reflecting any root with respect to the hyperplane corresponding to another root will again yield a root.

As we discussed before, the consecutive application of two reflections s_α and s_β, denoted by $s_\alpha \cdot s_\beta$, is a rotation and the set

$$W := \{s_{\alpha_{i_1}} \cdot s_{\alpha_{i_2}} \cdot \ldots \cdot s_{\alpha_{i_n}} \mid \alpha_{i_k} \in \Phi\}$$

of all possible combinations of reflections arising from elements of a root system is called the *Weyl group of* Φ.

For a given root system Φ, we are interested in a minimal subset that contains all the information about Φ.

Definition 3.2. A subset $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ of a root system Φ is called a *simple system* if

1. every linear combination of elements of Φ can be written uniquely as a sum $a_1\alpha_1 + \ldots + a_r\alpha_r$ where $a_1, \ldots, a_n$ are real numbers; we say that Δ is a vector space basis for the $\mathbb{R}$-span of Φ in $\mathbb{R}^n$, and
2. each $\alpha \in \Phi$ can be written as a sum $a_1\alpha_1 + \ldots + a_r\alpha_r$ where the $a_1, \ldots, a_n$ are real numbers and all of the same sign (all non-negative or all non-positive).

The elements $\alpha_1, \ldots, \alpha_r$ of a simple system are called *simple roots*.

Note that for a given root system, there is usually more than one possibility to choose the simple roots.

Example 3.3. We want to determine the root system associated to the graph C_3 (see Example 2.4). From the graph, we can compute that

$$\Delta = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\}$$

is a possible simple system. This will become clear after the definition of a Coxeter graph of a root system. To find the other roots to complete Δ to a root system Φ, we can use the properties (R1) and (R2) in Definition 3.1, for instance. By this, we obtain

$$\Phi = \left\{ \pm\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \pm\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \pm\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \pm\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \pm\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \pm\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \pm\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \pm\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \pm\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\}.$$

The root system Φ is pictured in Figure 5 and the choice of simple roots Δ is colored red.

We are now able to define the Coxeter graph of a root system.

Definition 3.4. Let Φ be a root system and let Δ be a simple system of Φ. We define the *Coxeter graph of* Φ *with respect to* Δ as follows:

The vertices are given by the elements of Δ. We connect two vertices corresponding to the simple roots $\alpha, \beta \in \Delta$ by an edge, if the rotation $s_\alpha \cdot s_\beta$ has order at least 3, and we label the edge with this order. Since the order 3 appears very often, we omit the label in this case.

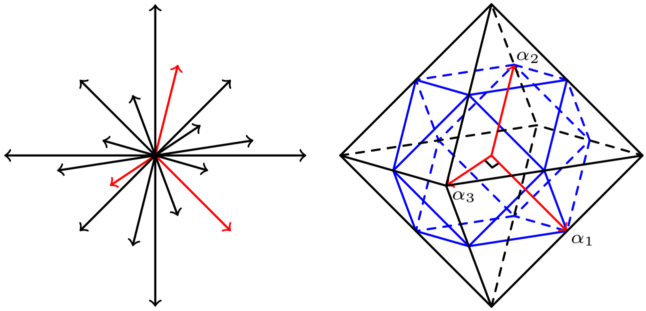


Figure 5: *Left:* Root system associated to the graph C_3. A possible choice of a simple system is colored red. *Right:* The octahedron associated to the root system on the left.

Now it shows what the condition (R2) in the definition of a root system is useful for: It makes sure that the order of any rotation $s_\alpha \cdot s_\beta$ is a natural number. If there is no edge between α and β, we interpret this as that the order of $s_\alpha \cdot s_\beta$ is 2. This makes sense since order 1 would imply that the rotation is the identity and hence $\alpha = \pm\beta$. But a rotation of order 2 is a rotation of angle 180° or π which is a reflection. In other words, α and β are orthogonal.

It is an interesting fact that, although the simple system Δ appears in the definition of the Coxeter graph, we actually obtain the same Coxeter graph if we use a different simple system. For more details, we refer to [4, Section 1].

Definition 3.5. We call a root system Φ *irreducible* if there do not exist two disjoint root systems Φ_1, Φ_2 such that Φ is the union of Φ_1 and Φ_2. We call it *reducible* otherwise.

With theory from linear algebra, we could show that irreducibility of root systems can be checked by studying the Coxeter graphs, as the following proposition details.

Proposition 3.6. *A root system Φ is irreducible if and only if its Coxeter graph is connected, that is, if any two vertices of the Coxeter graph are connected by a sequence of edges.*

Moreover, if Φ is reducible and W_1 and W_2 are the Weyl groups of Φ_1 and Φ_2, the Weyl group W of Φ can be constructed from W_1 and W_2, see [4, Prop. 2.2]. Thus, to understand a reducible root system, it is enough to know the irreducible root systems or the corresponding Coxeter graphs. Coxeter graphs can be studied without referring to a root system or a Weyl group like

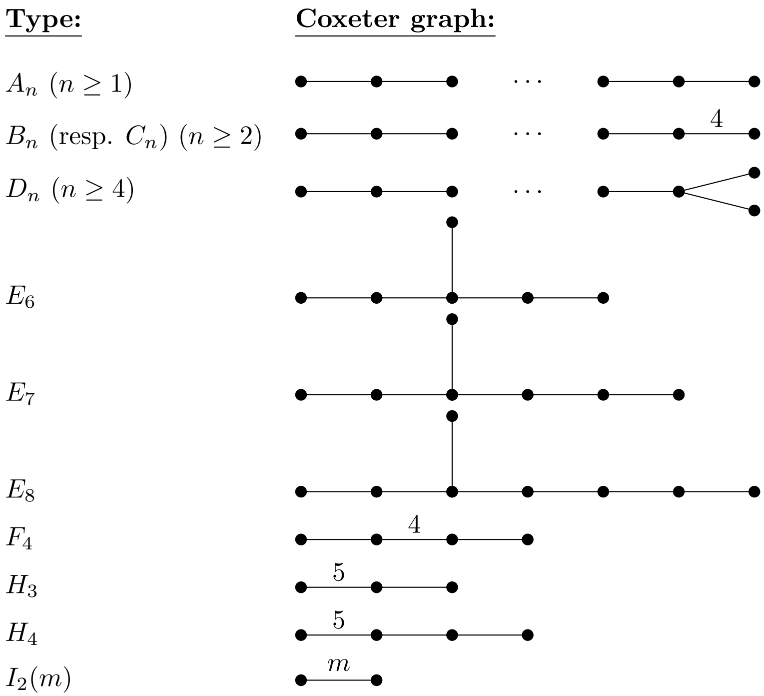


Figure 6: A complete list of all connected Coxeter graphs corresponding to irreducible Weyl groups [4, Section 2.4, Figure 1]. The index in the type symbolizes the dimension of the ambient space.

Humphreys does in [4, Section 2.3 ff.]. Moreover, Coxeter graphs are classified, that is, we know exactly which Coxeter graphs exist. In Figure 6, the reader can find a complete list of all connected Coxeter graphs corresponding to irreducible Weyl groups.

The classification of Weyl groups via Coxeter graphs (Figure 6) shows that there are only three possible Coxeter graphs with three vertices: A_3, B_3, and H_3. We already know from Example 2.4 that they correspond to the Platonic solids. A way to cook up the corresponding polyhedron is by using a "generalized Schläfli symbol", which can be found in [2, Chapter 20]. The graph can be translated into a Schläfli symbol, which can then be easily translated into a polyhedron. The Weyl group is, by construction, the corresponding *symmetry group* of the polyhedron, that is, the set of all transformations (for example, rotations) under which the polyhedron remains the same. Dual polyhedra share the same

Coxeter graph, read from the left or right side. Moreover, regular polyhedra come from linear Coxeter graphs, that means that all vertices of the graph lie on one line and do not branch off. In fact, there is a one-to-one correspondence between linear Coxeter graphs and regular polyhedra. Since Coxeter graphs are classified (see Figure 6), all regular polyhedra are classified in arbitrary dimension. In dimension 2, all Coxeter graphs are of type $I_2(m)$, symbolizing the regular m-gon. In dimension 3, we have the connected Coxeter graphs A_3, B_3 (also called C_3) and H_3, which are the already known Platonic solids from Figure 2. In dimension 4, we have the Coxeter graphs A_4, corresponding to the 4-*dimensional simplex*, B_4, corresponding to the *hypercube* (respective dual: 4-*orthoplex*), F_4, corresponding to the 24-*cell*, and H_4, corresponding to the 120-*cell* (respective dual: 600-cell). In dimension $n \geq 5$, the only linear Coxeter graphs are of type A_n and B_n, which are the n-*simplex* and the n-*cube* (respective dual: n-*orthoplex*).

4 Outlook and open questions

The Weyl group of a root system that we have seen in Definition 3.1 is an example of a "Coxeter group". In [5], Kazhdan and Lusztig introduce the *Kazhdan–Lusztig polynomials* of a Coxeter group W which consists of a family of polynomials, one polynomial for each pair of elements of W. Since then, mathematicians found significant applications across various fields of mathematics for Kazhdan–Lusztig polynomials but there are also a lot of open problems.

For instance, the "Combinatorial Invariance Conjecture" is one of the most prominent open problems. An article of Brenti gives in [1, Conjecture 1.4] a formulation of this conjecture and presents more open problems. For another modern application of root systems, see, for example, the article "The magic of 8 and 24" of Okounkov [6] which is a popular science article about the work of Maryna Viazovska who won the Fields medal in 2022.

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