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Convex polytopes and linear programs

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Convex polytopes are geometric objects that look deceptively simple. They occur everywhere in mathematics and have practical applications in everyday life – like organizing your grocery shopping list. In this snapshot, you get into contact with a long-standing, unsolved question in mathematics, which you can explore interactively.

1 The supermarket (linearly optimized)

Imagine that you are hungry, standing in front of a supermarket. You want to decide what to buy, taking into account basic recommendations for food consumption. Let us look at a very small supermarket, which only sells carrots and chocolate. Suppose 100 g of those carrots cost €0.12, while 100 g of chocolate cost €2.00. Then, for instance, you could buy 200 g of carrots and 300 g of chocolate for a total of €6.24. We can visualize the content of your shopping bag as a point with coordinates (x, y) in the plane, where $x = 2$ tracks the amount of carrots, and $y = 3$ measures the chocolate (in multiples of 100 g); see the gray point in Figure 1.

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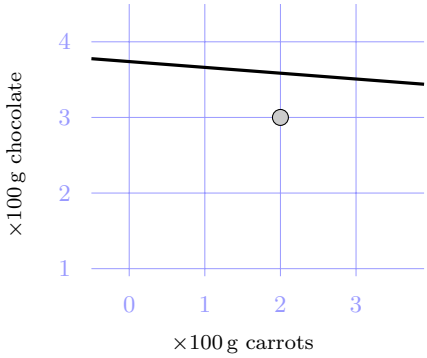

Figure 1: A shopping bag consisting of 200 g of carrots and 300 g of chocolate is represented by the gray point. Combinations below the black line correspond to less than 2 000 kcal energy.

Now, we need to think about energy. An amount of 100 g of carrots contains about 41 kcal of energy while 100 g of chocolate contains much more, namely about 535 kcal. Imagine we want to keep the total energy intake at the recommended amount of at most 2 000 kcal per day. In order not to exceed this constraint, a feasible shopping bag (x, y) needs to satisfy the *linear inequality* $41x + 535y \leq 2000$. In Figure 1, the constraint is visualized by the black line. Every point which lies below the black line corresponds to a shopping bag that does not contain too much energy. Our first shopping bag, containing 200 g of carrots and 300 g of chocolate, is one example.

Energy is not the only relevant factor, though. Imagine we also want to keep protein below 70 g, carbohydrates above 125 g, and vitamin A between 190 and 3 000 µg. Like the initial restriction to only two goods in the supermarket, this new restriction to just four rather randomly chosen nutritional components is totally artificial. However, it is just good enough to get the idea. Later we will talk about more realistic settings. So, for now, we are stuck with two goods,

Food Item	**Energy** (kcal)	**Protein** (g)	**Carbohydrates** (g)	**Vitamin A** (µg)
Carrots	41	1	10	835
Chocolate	535	8	59	59

Table 1: Nutritional values of 100 g of carrots and 100 g of chocolate according to a website of the U.S. Department of Agriculture [7].

carrots and chocolate, and four nutrients, energy, protein, carbs, and vitamin A. The nutritional values are listed in Table 1.

All four nutrients now impose constraints. Since vitamin A is bounded from above and below, we obtain five linear inequalities. In fact, we also need to take into account that buying negative amounts is not possible, so we additionally have $x \geq 0$ and $y \geq 0$, arriving at seven linear inequalities altogether. For simplicity, we assume that our two commodities can be acquired in arbitrary quantities, not necessarily exact multiples of 100 g. In this sense, our supermarket works a bit more like a farmer's market. The shopping bags that satisfy all our constraints simultaneously correspond to points in the gray area in Figure 2, which is called the *feasible set*.

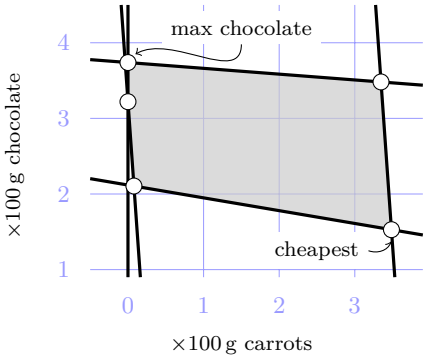

Figure 2: The constraints of all four nutrients restrict possible shopping bags to the feasible set, which is a convex polytope. Only five constraint lines are visible, since the other two lie outside the picture. The white points are the vertices of the polytope, they correspond to unique optimal solutions to certain goals.

Now, we need to decide on our goals. Imagine you want to follow the advice on nutrition, but as cheaply as possible. This task is a standard problem in mathematical optimization, known as a *linear program*:

minimize	$12x + 200y$	(total price)
subject to	$41x + 535y \leq 2000$	(energy $\leq$ 2 000 kcal)
	$x + 8y \leq 70$	(protein $\leq$ 70 g)
	$10x + 59y \geq 125$	(carbohydrates $\geq$ 125 g)
	$190 \leq 835x + 59y \leq 3000$	(190 µg $\leq$ vit. A $\leq$ 3 000 µg)
	$x, y \geq 0$	

In this case, it turns out that the best solution is to buy about 348 g of carrots and 153 g of chocolate, at a total price of €3.47. Figure 2 shows the

optimal solution as the white point at the lower right. This optimal solution is unique, which means that any other feasible shopping bag is strictly more expensive.

Sometimes the goal is to maximize some other objective; this does not make much of a difference from a conceptual point of view. For instance, if maximizing chocolate is all that matters, then the best choice is to buy about 374 g of chocolate for €7.48, no carrots in this shopping bag. That second linear program gives us a different white point as the unique optimal point in the feasible set.

The possible outcomes for various linear programs with a fixed set of constraints depend on the shape of the feasible set. A shape that is defined by lines this way is called a *convex polyhedron*. Those points which occur as unique optimal solutions for some objective function are the *vertices*. In our running example, there are five of them; these are the five white points in Figure 2.

Sometimes there are several optimal solutions for a linear program. For instance, if the objective is to maximize vitamin A, then all the feasible points on the near vertical line on the right are optimal. In Figure 2, the feasible set is a pentagon, which is the convex hull of its five vertices. It has five *facets*, which are those constraints which are not redundant. That is, there are two out of our seven linear inequalities which, in fact, do not constrain the feasible set. These are the bound on protein, $x + 8y \leq 70$, and the fact that you cannot buy negative amounts of chocolate, $y \geq 0$.

Our linear program is a little bit special in the sense that the feasible region is bounded, in which case the convex polyhedron is called a *convex polytope*. In our two-dimensional case, it is also called a *convex polygon*. Boundedness means that there are no feasible points with coordinates of arbitrarily large absolute value. These (convex) polytopes are the objects that we are actually interested in.

The analysis above crucially relies on the assumption that it is possible to buy any amount of carrots and chocolate in our supermarket. The problem becomes more complicated if the goods on sale are only available in integer multiples of 100 g. When this integrality condition has to be satisfied, the linear program is called an *integer* linear program. Figure 3 shows the six feasible and integral shopping bags as gray points. Solving an integer linear program is harder than solving a linear program. Yet it is possible to solve integer linear programs via a sequence of suitably chosen linear programs.

Before we look at more polytopes and their properties, it is now useful to reflect on our supermarket scenario. The reason why we could study all the linear programs so far in terms of two-dimensional pictures is the fact that we considered only two goods, carrots and chocolate. With a third commodity, such as apples, the same kind of analysis as before applies, but we get three-

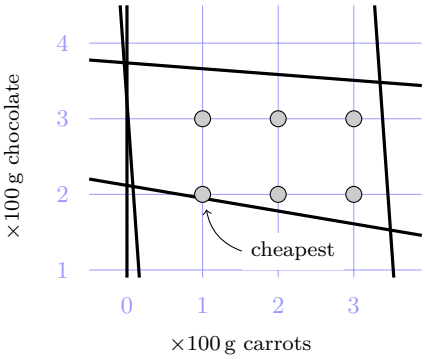

Figure 3: The gray points correspond to integer solutions. With this constraint, the cheapest solution is 100 g of carrots and 200 g of chocolate.

dimensional images. A typical supermarket has hundreds or thousands of goods on sale, so optimal shopping lists for larger supermarkets can be found by studying polytopes of rather high dimensions.

In the next section, we take a closer look at the geometry of three-dimensional convex polytopes. These polytopes can be thought of as coming from a linear program with three ingredients, and they are now bounded by flat planes, instead of lines as in our previous example.

2 Unfolding three-dimensional polytopes

Many intriguing questions in polytope theory and linear optimization are still unanswered. Most of these problems cannot be explained in this snapshot, because they require too much background. However, the descriptions of some open questions are quite accessible. Here we will look into one of them. As a caveat, this does not mean that the problem is easy to solve.

Albrecht Dürer is probably one of the most famous European artists of all times. He was born in Nürnberg, Germany in 1471 and he also died there, in 1528. Dürer was a painter and printmaker, and apparently he was interested in just about everything that came his way. His interest in drawing, etching and painting lead him to study perspective and the associated mathematics in a systematic way. Among his most advanced works in this area are descriptions of intricate constructions of planar and spatial objects [1]. Many of these objects are three-dimensional convex polytopes (which we will now call 3-*polytopes* for short), most notably the Platonic solids. Dürer wanted to give descriptions which were precise but could nonetheless be printed on a flat piece of paper. So

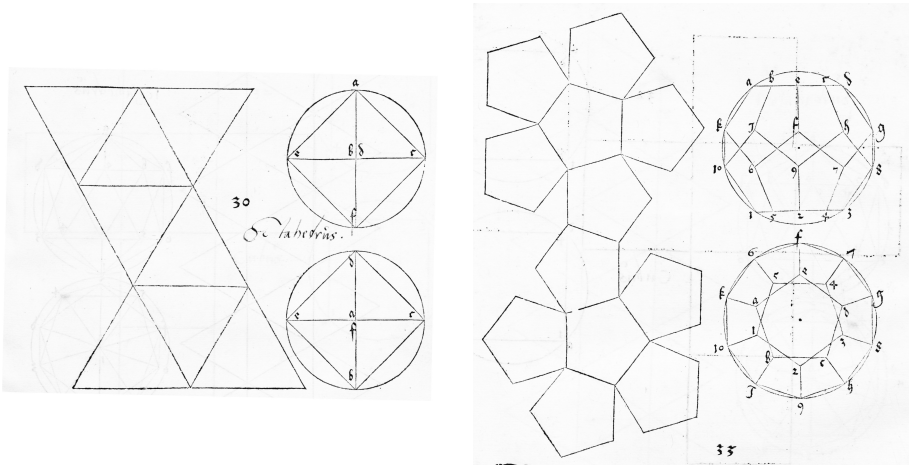

Figure 4: The octahedron and the dodecahedron with their nets drawn by Dürer [1].

he came up with exact planar drawings that are beautiful to look at; Figure 4 shows Dürer's renderings of the octahedron and the dodecahedron.

In modern terms, Dürer drew the *nets* of his polytopes. These arise from opening a 3-polytope by cutting along some of the edges such that the (still connected) uncut part can be unfolded to the plane, without overlaps. Such a net of a 3-polytope is practically useful: by cutting the net out and gluing along the missing edges, the polytope can be reconstructed, without any ambiguity.

Dürer's approach of constructing 3-polytopes from their nets leads to the obvious question: does this always work? That is to say, can any given 3-polytope be cut along edges such that the net does not have any overlaps in the plane? In full generality, this problem is unsolved. However, we have partial information.

The farthest reaching result known so far is a theorem by Mohammad Ghomi [3]. In 2014, he showed that every 3-polytope can be stretched, making it very long and thin, such that it has a non-overlapping net. It is worth noting that the stretching does not change the number of facets, or how the facets are connected to each other. So, knowing only which facets go together (without knowing their exact geometric shape) is not enough to determine whether a 3-polytope has a non-overlapping net.

Other results concern special classes of polytopes. For instance, Radons recently proved that nested prismatoids always admit a net [9]. A 3-polytope is called a *prismatoid* if it has two facets in parallel planes such that each vertex is contained in one of these facets. Prismatoids look a bit like drums. A

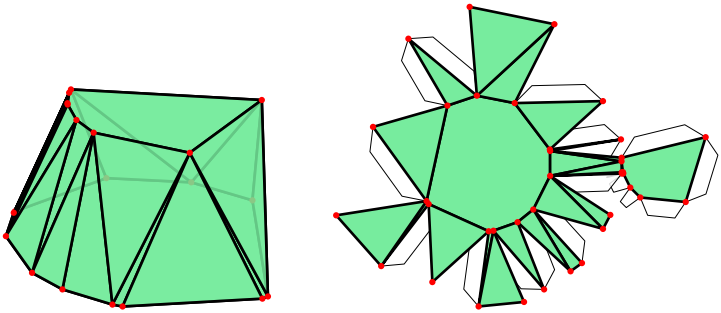

Figure 5: A nested prismatoid with a net. The white flaps can be used for gluing at the cut edges to actually build a three-dimensional model.

prismatoid is *nested* if viewed from far above, the top special facet fits within the lower parallel facet. Figure 5 shows a nested prismatoid and one of its nets.

The world of 3-polytopes is quite rich; playing with their nets is one way to get acquainted with them. To put this into practice, we created the educational game `MatchTheNet` [5], which can be played interactively in any web browser at `www.matchthenet.de`. This is a single-player game. In each round, some polytopes and their nets are shown. The player needs to match each net to the correct polytope. On the lowest level, the game is quite suitable for young children. However, higher levels are more challenging. The implementation of `MatchTheNet` has two components. The graphical interactive user front end is written in `JavaScript`, which is natively supported in web browsers. The polytopes and their nets are precomputed with `polymake` [2], which is a comprehensive open source software for research in polyhedral geometry.

3 Further Reading

Introductions to the theory of convex polytopes can be found in the textbooks of Grünbaum [4] and Ziegler [12]. Schrijver [10] explains linear programming and its generalizations. For a recent survey on nets and related topics, see O'Rourke and Vilcu [8].

It was a major breakthrough when Khachiyan found an algorithm for solving linear programs in polynomial time [6]. Nonetheless, the precise complexity status of linear optimization is still open [11, Problem 9]. This is only one of the remaining mysteries in the area which keep inspiring several generations of researchers.

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Image credits

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All other images created by the author with `polymake` [2] and `TikZ` [3].

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