Snapshots of modern mathematics
from Oberwolfach

№ 2/2023

The periodic tables of algebraic geometry

Pieter Belmans

To understand our world, we classify things. A famous example is the periodic table of elements, which describes the properties of all known chemical elements and gives us a classification of the building blocks we can use in physics, chemistry, and biology. In mathematics, and algebraic geometry in particular, there are many instances of similar "periodic tables", describing fundamental classification results. We will go on a tour of some of these.

To get a grip on the complexity of the world around us and the objects – such as animals, or chemical elements, or stars – appearing in it, we want to classify these objects. This allows us to describe the relationships, similarities, and differences between things we might be interested in.

An early, and somewhat cruel, effort to understand a class of living creatures lead to *lepidopterology*: the study of butterflies and moths, most famously performed by sticking needles through them and displaying them in nice wooden cases, such as in Figure 1. It serves as a prime example of classification in biology. Another important example is Darwin's description and classification of the beaks of the finches on the Galápagos islands, which led him to formulate the theory of evolution.

In this snapshot, I want to introduce you to the idea that *classification is an essential aspect of mathematics*, just like it is for biology (and other sciences). The mathematical objects that we will discuss are truly as pretty as

Figure 1: The work of a lepidopterist.

the butterflies from Figure 1. And whilst in some cases it takes a bit of training as a mathematician to fully grasp their beauty, at least no living creatures need to be harmed to study them.

In Section 1, we recall the *periodic table of elements*: an essential tool in modern chemistry, and the result of a lengthy classification effort. Luckily for us, the study of mathematical objects requires less interaction with dangerous chemicals.

In Section 2, we introduce the idea of classifications in algebraic geometry. These classification efforts are an ongoing process. When mathematicians complete one classification, they will move on to the next and more challenging one. That is why we will discuss "periodic tables in algebraic geometry", going from the 19th to the 21st century, and from completely known settings to cutting-edge research, in Sections 3 to 6.

1 The periodic table of elements

In every chemistry classroom, you find a large poster, listing the 118 known chemical elements together with their properties. This is the famous *periodic table*, and a very basic version is given in Figure 2.

The name periodic table refers to an experimentally observed periodicity in the chemical behavior of elements: certain elements tend to exhibit similar behavior. These observations are what chemists tried to formalise into a system. In 1869, the Russian chemist Dmitri Mendeleev (1834–1907) catalogued the then-known elements in terms of atomic mass, obtaining the periodic table we now know.

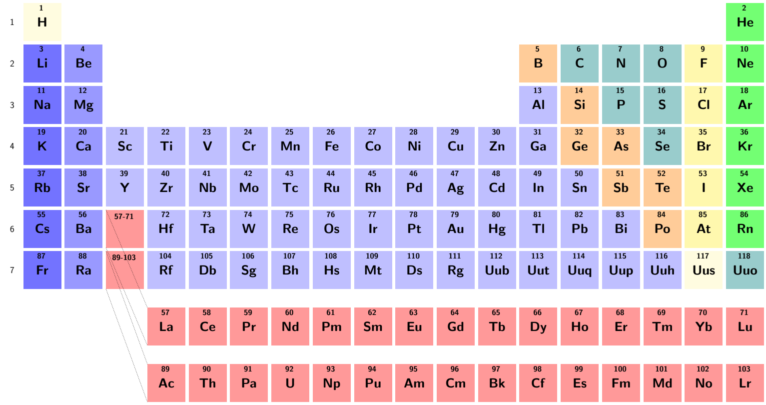


Figure 2: The periodic table of elements.

Originally there were gaps in the table: elements that were predicted to exist, but which had not yet been discovered. The periodicity of the periodic table also predicted some of the properties that these elements should have. For example, Mendeleev predicted the existence of an element with atomic mass ± 72.5, a high melting point, and a gray color. This element was subsequently found in 1887, and called *germanium*, in order to fill the gap which existed at position 32.

Invariants of elements The periodic table in Figure 2 lists only the chemical symbol and its atomic number. But usually a periodic table contains *lots* more data, such as the atomic weight, the melting and boiling point, the electron configuration, and so on. A beautiful interactive version can be found at `https://ptable.com`.

These are all examples of *invariants* of the objects being classified: properties of the chemical elements that do not change over time, and that do not depend on who measures them. By measuring invariants we can identify which chemical element we are looking at, and distinguish different elements. This is an important idea in mathematics too: mathematicians love to study invariants of objects, and then use them to distinguish between different objects.

Stars and the Hertzsprung–Russell diagram In the introduction, we also mentioned that one can try to classify the stars in the sky. To better understand an important aspect of classifications in mathematics it will help to discuss how classifying stars is different from classifying chemical elements.

Astronomers observed that not all stars are equal: some are brighter than others (even when accounting for their distance from Earth), and some are hotter than others. Back in the early 1910s, the Danish astronomer Ejnar Hertzsprung (1873–1967) and the American astronomer Henry Norris Russell (1877–1957) made a plot of those two properties of stars, and they noticed that some types of stars are impossible. There are no super-bright cold stars, nor are there very faint hot stars. And there are many stars like our Sun: they all have roughly the same brightness and the same temperature. The interested reader is invited to read up more on the Hertzsprung–Russell diagram.

The main takeaway is that it is possible to *vary the parameters* of a star, subject to certain rules imposed by physics. This behaviour is not present in the periodic table, but we will keep it in mind, as something similar will happen in mathematics.

2 Classifications in algebraic geometry

We now turn to classifications in the area of mathematics known as algebraic geometry. *Algebraic geometry* is the study of shapes described by polynomial equations. The shapes we are interested in are *smooth projective varieties*, defined over the complex numbers. Let us unpack what this means.

Smooth projective varieties First of all, we work with the *complex numbers* $\mathbb{C}$. Any complex number $z \in \mathbb{C}$ can be written in the form $z = x + \mathrm{i}y$ where x and y are real numbers and i is a square root of -1. Working over the complex numbers is necessary to make things tractable, but it also makes it harder to make drawings. Usually we visualise the complex numbers as the complex plane $\mathbb{R}^2$, with one real axis and one imaginary axis. But from the point-of-view of an algebraic geometer, the complex numbers are really a one-dimensional object! More concretely, Figure 4a is what an algebraic geometer would draw when drawing a curve, whilst Figure 5b is what a complex geometer would draw, but they really are manifestations of the same object.

Now, what does it mean to describe a shape *using polynomials*? A *polynomial* $f \in \mathbb{C}[x]$ over the complex numbers is an expression

$$f(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0$$

where the coefficients $a_0, \ldots, a_d$ are complex numbers. If d is the highest power appearing with a non-zero coefficient, we say that the polynomial f has *degree* d. We can substitute the variable x in the polynomial for any complex number and calculate the value of the expression. If $f(\alpha) = 0$, we call α a *zero* of f. Now we define the *variety* $\mathbb{V}(f)$ associated to f to be the set of all zeroes of f. We can express this definition formally by

$$\mathbb{V}(f) = \{\alpha \in \mathbb{C} \mid f(\alpha) = 0\}.$$

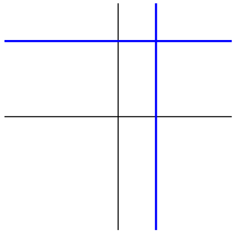


Figure 3: The zero set of the polynomial $f(x, y) = (x - 1)(y - 2)$.

Unless the polynomial is just $f(x) = 0$ and hence equal to 0 for any x, the set $\mathbb{V}(f)$ consists of at most d points by the Fundamental Theorem of Algebra. In particular, $\mathbb{V}(f)$ is a set with finitely many elements. As an example, for a quadratic polynomial $f(x) = ax^2 + bx + c$, the associated variety is $\mathbb{V}(f) = \{\frac{-b+\sqrt{b^2-4ac}}{2a}, \frac{-b-\sqrt{b^2-4ac}}{2a}\}$.

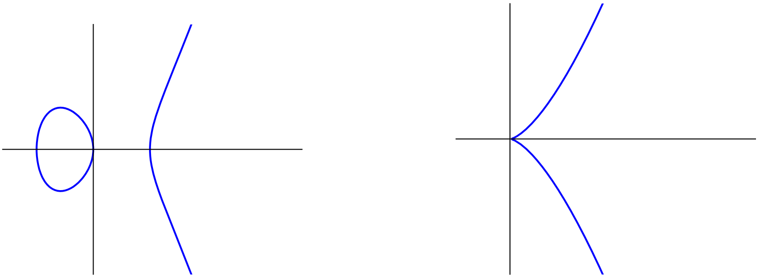
In general, we consider polynomials not only in one variable x but in n variables $x_1, \ldots, x_n$. We write $f \in \mathbb{C}[x_1, \ldots, x_n]$ and define again the associated *variety* to be the set of zeroes of f. This reads as

$$\mathbb{V}(f) = \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n \mid f(\alpha_1, \ldots, \alpha_n) = 0\}.$$

This is usually no longer a finite set, giving rise to much more interesting geometric objects! As an example, we can look at the polynomial $f(x, y) = xy$. The associated variety $\mathbb{V}(f)$ consists of all pairs of points (x, y) where $x = 0$ or $y = 0$. For a different example, see Figure 3.

Even more generally, we consider not only one but several polynomials and their zeroes at once, that is, the associated variety is the set of points that are zeroes of each of the polynomials simultaneously. We call $\mathbb{C}^n$ the *affine space* and these zero sets are called *affine varieties*.

In the affine plane $\mathbb{C}^2$, though, we have an annoying phenomenon: We would like to say that two distinct lines intersect in precisely one point. However, this is only true if the two lines are not parallel and we have to always mention this exception of parallel lines to make a correct statement. That is why we replace our affine geometry by something called *projective geometry*: In the projective space, we consider additional points "at infinity" and we say that two parallel lines intersect in such a point at infinity, so any two distinct lines now always intersect without exception. Hence, instead of affine varieties, we will consider *projective varieties*. This also means that we have to change the polynomials that we consider a bit but we will sweep that under the rug.



(a) Real points of the cubic curve $y^2 = x^3 - 2x$.

(b) Real points of the singular cubic curve $y^2 = x^3$.

Figure 4: A smooth versus a singular cubic curve.

The final ingredient in order to describe the objects we are interested in is *smoothness*. This is best explained through an example: consider the following polynomials of degree 3

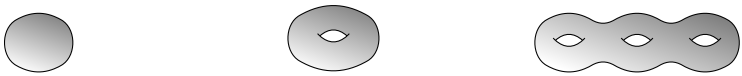
$$f = -y^2 + x^3 - 2x$$
$$g = -y^2 + x^3 + x^2$$

which describe affine curves in $\mathbb{C}^2$. If we draw these curves inside $\mathbb{R}^2 \subset \mathbb{C}^2$, we get the pictures as in Figure 4. We immediately see that on the right, there is something funny happening at the origin: there is a *singularity*. We call a curve *smooth* if it does not have a singularity. For more on singularities, we refer to another snapshot [1].

Classifying smooth projective varieties? In what follows next, we discuss examples of classifications of smooth projective varieties. This will illustrate how the life of an algebraic geometer can be very similar to that of someone sticking needles through unsuspecting butterflies, or that of an experimental chemist inhaling noxious fumes in order to isolate an unknown chemical element.

Before we embark on our journey, we need to point out that to an algebraic geometer classification can mean different things. Certainly, we are not just classifying polynomials, rather we are interested in classifying varieties independently of their realisation. For example, the polynomials $f(x,y) = x^2y$ and $g(x,y) = xy^2$ are different but the associated affine varieties are the same, namely the set of all pairs of points (x,y) where $x = 0$ or $y = 0$. This gives rise to the classification we will mostly be talking about: that of varieties *up to isomorphism*, that is, up to their realisation inside some projective space.

For one-dimensional objects, or curves, we can come up with a reasonably complete classification, see Section 3. But it turns out that already for two-



(a) A curve of genus 0. (b) A curve of genus 1. (c) A curve of genus 3.

Figure 5: Complex algebraic curves as Riemann surfaces.

dimensional varieties, the full classification is an impossible task, so we restrict ourselves to classifying certain well-chosen objects. There is an entire branch of algebraic geometry, called birational geometry, devoted to understanding the precise relationships between smooth projective varieties which are different but nevertheless almost the same; but we will not discuss this further.

3 Curves: Riemann

The first classification we look at is of the simplest objects we can try to classify: curves, or one-dimensional varieties.

We have already seen an important recipe and some examples to describe curves: take a polynomial $f \in \mathbb{C}[x, y]$, consider its projective version, and determine the associated variety $\mathbb{V}(f)$ (see for example Figure 4).

Can we describe all curves using the projective version of a single polynomial in two variables? In other words, is every curve a *plane curve*? It turns out that the answer is no. But we can show that every curve is a *space curve*: it can be described by the projective version of several polynomials in three variables, so that it is a curve in a three-dimensional space.

But does this help in the classification of curves? Recall that we don't want to classify polynomials, because those are only the tools to describe the curves. So we need to talk about things which are *intrinsic* to the geometry of a curve, independent of the realisation.

A discrete invariant For this, we can look at an algebraic curve as a so-called Riemann surface: We mentioned already that an algebraic geometer thinks of a curve as a one-dimensional object over the complex numbers whereas a topologist uses the description of the complex numbers $\mathbb{C}$ as the plane $\mathbb{R}^2$. Hence for them, a curve is a two-dimensional object over the real numbers $\mathbb{R}$. This is also how they were originally studied by Riemann in the 1850s. We end up with drawings as in Figure 5. These drawings correspond to plane curves of degree 2, 3 and 4 respectively. The obvious difference between these pictures is "the number of holes". Mathematicians refer to this number as the *genus*.

It is similar to what a biologist would use to distinguish a giraffe from a fish: the former has four legs, the latter has none. Therefore they must be different

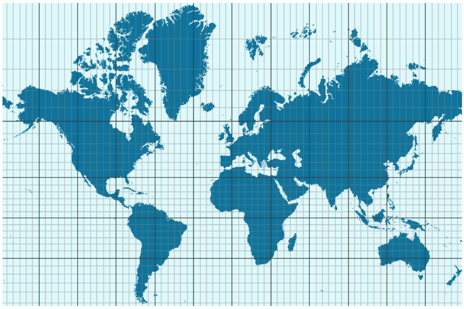


Figure 6: Earth is *positively curved*: Greenland seems to have the same size as Africa in the Mercator projection, but in reality is 15 times smaller.

in a meaningful sense, and the biologist will say they belong to different species (even though they are both animals).

So for now the "periodic table" of the classification of curves looks pretty bland: it is just the sequence of integers $0, 1, 2, \ldots$ But biologists don't stop classifying animals after having counted their legs, and neither will we after counting holes.

On a Riemann surface, we can additionally study the *curvature*. It measures how straight lines on the surface have to bend. We can distinguish the cases where the curvature is positive, where it is 0, and where it is negative. The case of a sphere (as in Figure 5a) has positive curvature. We are in fact familiar with an important consequence of this! If we think of the sphere that is our world, we know that any projection (such as the Mercator projection in Figure 6) will distort reality: it is impossible to make a flat map of the entire planet, because Earth has (positive) curvature.

The surface from Figure 5b has zero curvature (we say it is *flat*), and in all other cases (such as in Figure 5c), the surfaces have negative curvature. Thus, there is an additional *trichotomy* into $g = 0$, $g = 1$ and $g \geq 2$.

Continuous parametrisations But this is still not the end of the classification of algebraic curves. We have described a Riemann surface of genus 1 using the projective version of a polynomial of degree 3. What happens if we start varying the coefficients of this polynomial? For example, what if instead of the equation $y^2 = x^3 - 2x$ from Figure 4a, we consider $y^2 = x^3 - 3x + 1$? We can show that they define "the same" curve: they are *isomorphic*.

But what if we consider $y^2 = x^3 - 2x + 1$ instead of $y^2 = x^3 - 2x$? Then we can actually show that the curves are different, even though the genus is 1 in both cases. For this we'd have to use something called the j-invariant: a number attached to every genus 1 curve. In the case of $y^2 = x^3 - 2x$, it is equal to 1728, in the case of $y^2 = x^3 - 2x + 1$, it is 11059.2 (in case you wonder: the j-invariant is not a count of anything, in general it can be any complex number). Thus to a trained algebraic geometer, these are in fact different curves. Hence we can use parameters in our equations and get truly distinct answers. In our example, we changed the constant term from 0 to 1, and we could in fact have considered all intermediate values (including complex numbers) to get all kinds of non-isomorphic curves of genus 1. They are usually referred to as *elliptic curves*, and they are an important tool in modern cryptography.

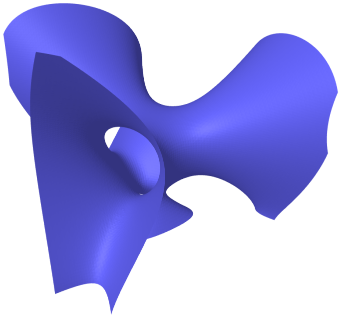
This continuous behavior is not something that happens in the periodic table of elements: you can't move from hydrogen to helium by adding tiny fractions of neutrons, electrons and protons. The closest analogy in science is the Hertzsprung–Russell diagram, where you can vary the luminosity and temperature of a star continuously.

What about other degrees? Here the trichotomy into $g = 0$, $g = 1$, and $g \geq 2$ comes back into play. We can show that for $g = 0$, there are no parameters possible (so there is a single curve of genus 0), whilst for $g \geq 2$, there are in fact $3g-3$ parameters (so there are *many* curves of genus g, and describing them all in a suitable sense is an interesting problem). This result for $g \geq 2$ is what Riemann obtained back in 1853, effectively introducing the notion of a *moduli space* to mathematics: a parameter space to describe all curves of a given genus.

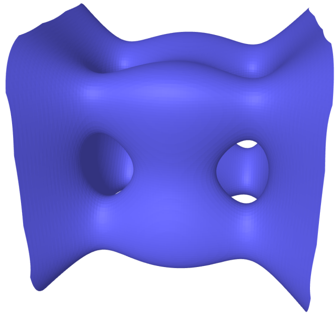
4 Smooth projective surfaces: Enriques

Going up one dimension, we end up with complex surfaces. They have been at the forefront of algebraic geometry since the 19th century. The easiest algebraic surfaces we can produce are by taking the projective version of a polynomial in three variables, and considering the associated variety in three-dimensional space. Because we are working over the complex numbers, this would require a four-dimensional drawing, which goes beyond what we can do here. But in Figure 7, we do what algebraic geometers often do: make a picture of an affine piece over the real numbers. The platform IMAGINARY in fact offers software to do this easily: `https://www.imaginary.org/program/surfer`.

The classification of smooth projective surfaces is due to the Italian mathematician Federigo Enriques (1871–1946), as his life's work, posthumously published in 1949. We necessarily have to gloss over many details, but we will highlight some of its most interesting features. Without going into a detailed explanation, we use that every surface can be reduced in a controlled way to what is called a "minimal" surface. Moreover, we also know how to go back from a minimal surface to our original one. Thus it suffices to classify these minimal surfaces.



(a) The Clebsch cubic surface.



(b) A quartic K3 surface.

Figure 7: Two algebraic surfaces considered in $\mathbb{R}^3$.

Trichotomy Just like for curves, we have a notion of curvature for complex surfaces, introducing an important trichotomy between positive, flat, and negative. With curves, we had that the *positively curved* case was the easiest, there being a unique such curve. For surfaces, this is still the easiest case, but the uniqueness no longer holds: there are now 10 families of what are called *del Pezzo surfaces*, named after the mathematician who classified them in 1887. Some of these are unique in their family, for others there are continuous parameters.

If we consider the projective version of a single polynomial in three variables, the cases of degree $d = 1, 2$ and 3 give rise to del Pezzo surfaces. In Figure 7a, we have given an impression of an important surface, where $d = 3$. The geometry of these del Pezzo surfaces is already rich enough to fill entire books – even though their classification is relatively straightforward – and their higher-dimensional analogues will be important for what follows.

What about the flat case, the two-dimensional analogue of elliptic curves? There are now *two* distinct families. The closest analogue of elliptic curves are *abelian surfaces*. But there are also *K3 surfaces*, named so by the French mathematician André Weil (1906–1998) in 1958 after the recently climbed K2 mountain in the Himalayas, and the three mathematicians Kummer, Kähler, and Kodaira, who had been building the tools to study algebraic geometry and these surfaces in particular.

If we again consider the projective version of a single polynomial in three variables, the case $d = 4$ gives rise to a K3 surface; in Figure 7b we see an impression of an example. As with del Pezzo surfaces, their geometry is rich enough to fill entire volumes, and their higher-dimensional analogues will again be important for what follows.

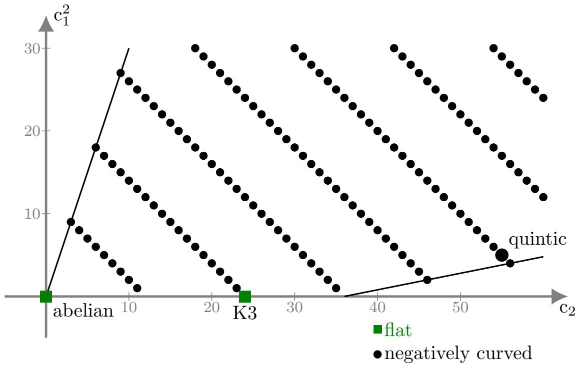


Figure 8: The geography of algebraic surfaces.

The geography of surfaces: surfaces of general type We now come to the analogue of curves of genus $g \geq 2$. There are such curves for every value of g, in fact there is an entire parameter space of them with $3g - 3$ parameters, describing their classification.

Unlike for curves, a single integer is no longer enough to describe the crude classification of surfaces. Two important integers we can assign to a surface are the *Chern numbers* c_1^2 and c_2. For the genus, we only had the inequality $g \geq 0$ because we were counting something. For surfaces, the situation is more complicated, and the possible values depend on the curvature. In Figure 8, we have drawn the "allowed" values for small Chern numbers. These conditions are similar to a law in biology saying that the number of legs on an animal is always even (but starfish are obvious counterexamples, so this particular universal biological law does not exist).

Now we have discussed *necessary* conditions on the Chern numbers. Are these also *sufficient* conditions, that is, can we always find a surface with these allowed numbers? This leads to the problem of understanding the *geography of surfaces*, in particular those with negative curvature, which are called *of general type*.

We can already fill in two positions in our "atlas" in Figure 8: abelian surfaces have $c_2 = 0$ and K3 surfaces have $c_2 = 24$. Next, if we take a polynomial of degree 5, we get a quintic surface, for which $c_1^2 = 5$ and $c_2 = 55$. There are still *many* other allowed values in Figure 8, and it is an interesting challenge to find a construction for a surface with given Chern numbers. We don't know yet whether every allowed pair corresponds to a surface, but we do know of *many* interesting and beautiful examples.

The important takeaway is that the classification of surfaces of general type is still an open problem, and it has been giving mathematicians enough material to work on for over a century.

5 Fano 3-folds: Mori–Mukai

In our journey through smooth projective varieties, we now reach dimension 3. From this point on, it is impossible to make good pictures (although algebraic geometers do develop an intuition for these objects, and make drawings which are hard, if not impossible, to interpret for outsiders).

For curves and surfaces, we saw that the trichotomy between positive, flat, and negative curvature gave very different flavours to the classification problem. This pattern continues in higher dimensions. The analogue of the $g = 0$ case for curves (with positive curvature) and del Pezzo surfaces are called *Fano varieties*, and in dimension 3 these are called *Fano 3-folds*. We will first talk about these, as a full classification indeed exists. In dimension 2, we already saw that there are 10 families of del Pezzo surfaces. So, what about dimension 3?

Again, there exists a full classification, due to the Japanese mathematicians Shigefumi Mori and Shigeru Mukai in 1981 (with important preliminary work by the Russian mathematician Vasilii Alekseevich Iskovskikh), building on the results for which Mori eventually won the Fields medal in 1990, one of the highest honors in mathematics. There are 105 families in the classification: originally they listed 104, but back in 2003 they found a missing case.

The geometry of Fano 3-folds is truly a treasure trove of interesting algebraic geometry, with lots of ongoing work which falls outside the scope of this snapshot. Having a classification of the objects is after all not the end of the work, but rather the beginning of the systematic study.

Calabi–Yau 3-folds We will now consider 3-dimensional varieties with flat curvature: *Calabi–Yau 3-folds*. They are the analogues of the K3 surfaces and abelian surfaces that we saw before. These objects have played a tremendously important role in theoretical physics and string theory, and given their importance, mathematicians have been constructing more and more of them. Their beautiful properties and ongoing classification would form an excellent subject for yet another snapshot.

But frustratingly enough, we don't know whether the final classification in this case is a finite classification or not! To give a precise number of currently known families of Calabi–Yau 3-folds is hard, because it requires a careful comparison of all the different constructions. Let us just point out that one important type of construction (using reflexive four-dimensional polytopes, of which there are a whopping 473 800 776) gives rise to 30 108 families of Calabi–Yau 3-folds which are guaranteed to be different.

6 Hyperkähler varieties

One important theme that we have seen is that the higher we go in dimension, the more restrictive we need our class of varieties to be in order to have any hope of classification.

Amongst the varieties of flat curvature, there exists a decomposition into building blocks, just like we can decompose molecules into atoms (for arbitrary varieties there is nothing like such a decomposition). There are 3 types:

- abelian varieties of arbitrary dimension;
- Calabi–Yau varieties of dimension ≥ 3;
- hyperkähler varieties.

So the classification problem of varieties with flat curvature splits into three different classification problems.

We already mentioned the classification of Calabi–Yau 3-folds, and in arbitrary dimension, the situation is the same: we don't know whether the classification is finite, but we can construct *many* (really, many!) examples. On the contrary, although they possess lots of interesting geometry, the classification of abelian varieties is straightforward: in every dimension, there is a single family.

That leaves us with hyperkähler varieties. These are necessarily even-dimensional, and possess an extremely rich and beautiful geometry.

The first examples of dimension ≥ 4 were obtained by the French mathematician Arnaud Beauville in 1983. Using K3 surfaces, he constructed a family of hyperkähler varieties of dimension $2n$. We call varieties of this type $\mathrm{K3}^{[n]}$. Similarly using abelian surfaces, he constructed another family of hyperkähler varieties of dimension $2n$. We call varieties of this type Kum^{n}.

In 1999 and 2003, Kieran O'Grady constructed two new families of hyperkähler varieties. In one of the families, the varieties are six-dimensional; in the other, they are 10-dimensional. We call the families OG_6 and OG_{10}, respectively. Currently they look "exceptional", in the sense that they are seemingly not part of a construction that works in arbitrary dimension.

Is *this* then the end of the classification? Are all hyperkähler varieties of type $\mathrm{K3}^{[n]}$, Kum^{n}, OG_6 and OG_{10}? We have **absolutely no idea**! Already in dimension 4, we don't know whether $\mathrm{K3}^{[2]}$ and Kum^{2} are all the types we need. There might be a type of hyperkähler variety that has been hiding from us, like a beautiful butterfly deep within the rain forest. In other words, we are still far from understanding the periodic table of hyperkähler varieties.

Interactive periodic tables in algebraic geometry Do you want to see some "periodic tables" in algebraic geometry in action? The author has created various interactive interfaces for some of the classification results:

- `https://mgnbar.info`: the geometry of the moduli space of curves
- `https://superficie.info`: Enriques–Kodaira classification of surfaces (joint with Johan Commelin)
- `https://fanography.info`: Mori–Mukai classification of Fano 3-folds
- `https://hyperkaehler.info`: classification of hyperkähler varieties
- `https://grassmannian.info`: generalised Grassmannians (not discussed)

It might be hard to really understand what is happening there, but hopefully it is clear that mathematicians are truly interested in classifications.

Acknowledgements I would like to thank the organisers of Oberwolfach workshop number 2228, Algebraic Geometry: Moduli Spaces, Birational Geometry and Derived Categories, for the invitation to prepare this snapshot. The work on this snapshot was supported by the Luxembourg National Research Fund (FNR) (PSP-Classic: Classified Mathematics).

Image credits

Figure 1 "Collection drawer with butterflies in Upper Silesian Museum in Bytom, Poland". Author: Marek Ślusarczyk. Licensed under Creative Commons Attribution 3.0 Unported, visited on January 13, 2023.

Figure 2 "Periodic table of elements". Original authors: Ryan Griffin and Janosh Riebesell. Licensed under MIT License, modified by the author from `https://tikz.net/periodic-table`, visited on January 13, 2023.

Figure 6 "Mercator projection". Author: Miaow Miaow. Licensed under Creative Commons Attribution-Share Alike 3.0 Unported, visited on January 13, 2023.

Figures 4, 5, 7 and 8 Created by the author using SageMath or Ti*k*Z.

References

[1] R. O. Buchweitz and E. Faber, *Swallowtail on the shore*, Snapshots of modern mathematics from Oberwolfach (2014), no. 7, https://doi.org/10.14760/SNAP-2014-007-EN.

Pieter Belmans *is a professor of mathematics at the University of Luxembourg.*

Mathematical subjects
Algebra and Number Theory, Geometry and Topology

Connections to other fields
Chemistry and Earth Science, Life Science, Physics

DOI
10.14760/SNAP-2023-002-EN

Snapshots of modern mathematics from Oberwolfach provide exciting insights into current mathematical research. They are written by participants in the scientific program of the Mathematisches Forschungsinstitut Oberwolfach (MFO). The snapshot project is designed to promote the understanding and appreciation of modern mathematics and mathematical research in the interested public worldwide. All snapshots are published in cooperation with the IMAGINARY platform and can be found on www.imaginary.org/snapshots and on www.mfo.de/snapshots.

ISSN 2626-1995
