Snapshots of modern mathematics
from Oberwolfach

№ 17/2015

From computer algorithms to quantum field theory: an introduction to operads

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An operad is an abstract mathematical tool encoding operations on specific mathematical structures. It finds applications in many areas of mathematics and related fields. This snapshot explains the concept of an operad and of an algebra over an operad, with a view towards a conjecture formulated by the mathematician Pierre Deligne. Deligne's (by now proven) conjecture also gives deep inights into mathematical physics.

1 Introduction

When solving a problem, it is usually important to understand its true nature and meaning. Imagine for example a jigsaw puzzle with all pieces turned upside down. You can still solve it using only the shapes of the pieces, but it will be harder and relatively boring, because you will only understand what it is all about when you flip the pieces over.

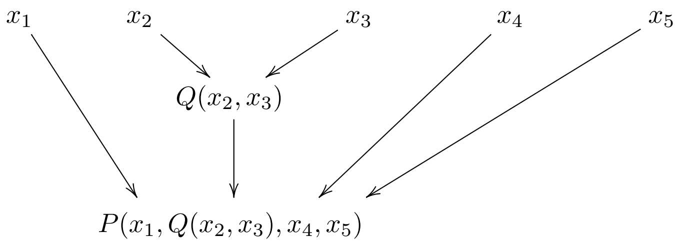
Furthermore, problems which at first sight appear to be different can turn out to be the same when viewed in an appropriate way; or they may at least be solved using the same tools. The present snapshot is about such a universal tool from mathematics and neighbouring fields. It is called the theory of *operads*.

2 An operad from computer science

2.1 The general idea

Given a specific type of computer, one can ask how many different functions (that is to say algorithms or programmes) P one can use it to compute. Let $\mathcal{P}_n$ denote the list of all these functions that take inputs $x_1, \ldots, x_n$ of a fixed type (for instance this could be the type "number") and spit out a single output $y = P(x_1, \ldots, x_n)$ of the same type.

Observe now there is an easy way of forming new functions out of old ones: if $P \in \mathcal{P}_n$ and $Q \in \mathcal{P}_m$ (the symbol "$\in$" means "is an element of", so P is a function with n inputs and Q is one with m inputs), then we can feed the output of Q as the i-th input into P, where i is any number from 1 to n. The result is a function with $m + n - 1$ inputs that we denote by $P \cdot_i Q$. Pictorially, we could denote this as follows:



In this example, $m = 2$, $n = 4$, $i = 2$, and $x_1, \ldots, x_5$ are the inputs of $P \cdot_2 Q$. In terms of an abstract formula, $P \cdot_i Q$ is generally defined by

$$\begin{aligned} &(P \cdot_i Q)(x_1, x_2, \ldots, x_{m+n-1}) \\ &= P(x_1, x_2, \ldots, x_{i-1}, \underbrace{Q(x_i, \ldots, x_{i+m-1})}_{i\text{-th input of } P}, x_{i+m}, \ldots, x_{m+n-1}). \end{aligned} \quad (1)$$

2.2 An example

Consider a simple computer that has two basic operations, the multiplication M and the addition A of natural numbers. We can think of these operations as functions with two inputs and one output and write

$$A(x, y) = x + y, \quad M(x, y) = x \cdot y.$$

In other words, A and M form the set $\mathcal{P}_2$. The set $\mathcal{P}_3$ then consists of all functions that we can build out of A and M using the products $\cdot_i$, such as for example the function $A \cdot_1 M$ which is defined by

$$(A \cdot_1 M)(x, y, z) = A(M(x, y), z) = x \cdot y + z.$$

In total, we can form eight such expressions,

$$A \cdot_1 A,\ A \cdot_1 M,\ M \cdot_1 A,\ M \cdot_1 M,\ A \cdot_2 A,\ A \cdot_2 M,\ M \cdot_2 A,\ M \cdot_2 M.$$

However, not all of them are different! Indeed, we have

$$A \cdot_1 A = A \cdot_2 A, \quad M \cdot_1 M = M \cdot_2 M \tag{2}$$

as you can verify by expanding the above expressions for arbitrary inputs. The reason is – as you might have found out – the *associativity* of addition and multiplication, meaning

$$(x + y) + z = x + (y + z), \quad (x \cdot y) \cdot z = x \cdot (y \cdot z)$$

holds for all natural numbers x, y, and z.

Quiz. Show that all the other expressions are different, so that $\mathcal{P}_3$ has exactly six elements. How many elements does $\mathcal{P}_4$ have? Can you find a general formula for the number of elements in $\mathcal{P}_n$?

2.3 Associativity of $\cdot_i$

At this point, we make another observation: $\cdot_1$ itself is associative, that is,

$$(P \cdot_1 Q) \cdot_1 R = P \cdot_1 (Q \cdot_1 R)$$

holds for all functions P, Q, R. For $i \neq 1$ it gets more complicated, though. For example, if $P \in \mathcal{P}_3$, $Q \in \mathcal{P}_2$ and $R \in \mathcal{P}_1$, then

$$(P \cdot_2 Q) \cdot_2 R = P \cdot_2 (Q \cdot_1 R)$$

holds: both sides equal the function $S \in \mathcal{P}_4$ which is given by

$$S(x, y, z, t) = P(x, Q(R(y), z), t).$$

Looking a bit closer at these products $(P \cdot_i Q) \cdot_j R$, one finds out that the sets $\mathcal{P}_n$ reveal an interesting structure. Namely, for any $P \in \mathcal{P}_n, Q \in \mathcal{P}_m, R \in \mathcal{P}_l$ and for all computers and types of input they satisfy a set of rules:

$$(P \cdot_i Q) \cdot_j R = \begin{cases} (P \cdot_j R) \cdot_{i+l-1} Q & \text{if } j < i, \\ P \cdot_i (Q \cdot_{j-i+1} R) & \text{if } j = i, \\ (P \cdot_{j-m+1} R) \cdot_i Q & \text{if } j > i. \end{cases} \tag{3}$$

You can verify these rules by using the abstract product definition given in (1).

2.4 Operads

Whenever mathematicians find a structure interesting, they turn it into an abstract definition. In case of the $\mathcal{P}_n$'s and of what we have studied so far, this results in the definition of an *operad*:

Definition. An *operad* $\mathcal{P}$ is a sequence of sets $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \ldots$ together with products $P \cdot_1 Q, \ldots, P \cdot_n Q$ in $\mathcal{P}_{m+n-1}$ for all $P \in \mathcal{P}_n$ and $Q \in \mathcal{P}_m$ that satisfy the set of rules (3) above.

An example of an operad is, as we have seen, the sequence of $\mathcal{P}_n$'s introduced in Section 2.1. Other examples will follow soon.

2.5 Symmetric operads

So far, we have seen that the product $\cdot_i$ combining functions from $\mathcal{P}_n$ and $\mathcal{P}_m$, respectively, produces a new function from $\mathcal{P}_{m+n-1}$. There is another way of creating new functions: if we *permute* (meaning we shuffle or reorder) the inputs of a function, we can produce new functions. For example, let $\mathcal{P}_n$ be the simple addition-multiplication operad from Section 2.2 and let $\mathcal{Q}_n$ be obtained by permutation of the inputs.

To begin with, we have $\mathcal{Q}_2 = \mathcal{P}_2$, because it is

$$A(x, y) = x + y = y + x = A(y, x)$$

and similarly $M(x, y) = M(y, x)$. In mathematical language we say both addition A and multiplication M are *commutative* operations, that is swapping the two variables does not create a new function. However, this is no longer true for $n > 2$:

Quiz. Show that P given by $P(x, y, z) = (A \cdot_1 M)(y, x, z) = x \cdot z + y$ is not in $\mathcal{P}_3$. Show that $\mathcal{Q}_3$ contains eight elements.

In the theory of operads, there are objects we call *symmetric operads.* These are operads with a generalisation of the input permutation operation above. The interested reader can find more information on these in [5].

2.6 One more: the endomorphism operad

Moving now slowly from computer science to pure mathematics, we can note that any set X whatsoever defines an operad $\mathcal{E}(X)$. Namely, $\mathcal{E}(X)_n$ consists of all functions f with inputs $x_1, \ldots, x_n \in X$ and one output $f(x_1, \ldots, x_n) \in X$.

Definition. $\mathcal{E}(X)$ is called the *endomorphism operad* of X.

In other words, $\mathcal{E}(X)$ is the operad of the universal computer in which one can implement any function of variables and with values in X.

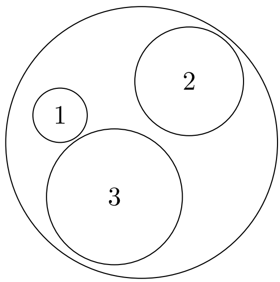
Quiz. If X has 3 elements, how many elements does $\mathcal{E}(X)_n$ have?

Thus, by definition, the operads $\mathcal{P}$ defined earlier were all subsets of $\mathcal{E}(X)$, where X is the set of all values that variables of the functions in $\mathcal{P}$ can take. We will now see an intriguing operad that is not of this form.

3 Little discs and the Deligne conjecture

3.1 The little discs operad $\mathcal{D}$

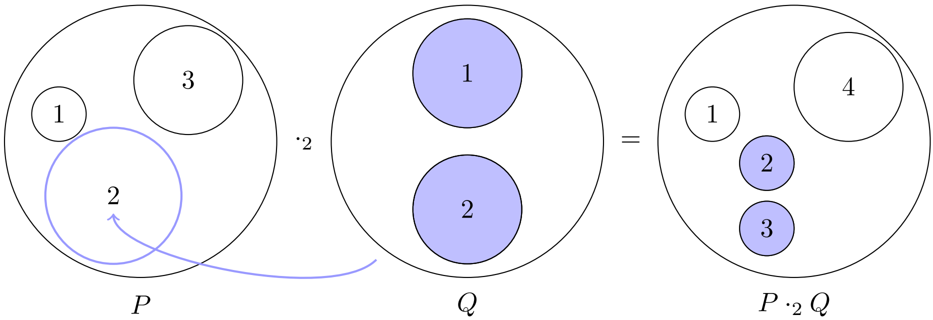
A *configuration of n little discs* specifies the positions and radii of n numbered non-intersecting discs inside a single disc of radius 1. The figure below shows a configuration of $n = 3$ little discs:



Let us denote the set of all possible such pictures by $\mathcal{D}$. Then $\mathcal{D}$ becomes an operad if, for given P, $Q \in \mathcal{D}$, we define the products $P \cdot_i Q$ as the result of the following procedure:

Product algorithm: Glue a scaled copy of the picture Q into the i-th disc in P and discard the boundary of this i-th disc subsequently. Finally, renumber the discs, starting with the first disc in P and numbering the newly inserted discs in the same order as they appeared in Q, starting with i.

The following picture illustrates the process for $i = 2$:



Here, the discs (1) and (3) of P are the discs (1) and (4) in $P \cdot_2 Q$. Disc (2) gets replaced by a scaled copy of Q, so the discs (1) and (2) in Q become discs (2) and (3) in $P \cdot_2 Q$. I hope this explains the recipe sufficiently. One can check that this product satisfies the rules (3) which makes $\mathcal{D}$ an operad. We call it the *little discs operad*. Moreover, noticing that the little discs within a configuration can be permuted, you maybe can even believe that $\mathcal{D}$ is also a symmetric operad.

Quiz. In Section 2.2, we found out that the associativity of the elements M and A of $\mathcal{P}_2$ implies (2). Show that there is no configuration $P \in \mathcal{D}_2$ of two little discs which is associative (that is, you have to show that there is none that satisfies $P \cdot_1 P = P \cdot_2 P$).

3.2 Kontsevich and Deligne

Changing gears, imagine a balloon filled with gas. By pressing and stretching you can deform it into many shapes, but, for example, not into a doughnut-shaped object, which in mathematics one usually calls a *torus*. With this picture in mind, we can easily imagine such deformations of geometrical objects. In mathematics, one describes and classifies these on a formal level and also deals with the "deforming" of more abstract structures (of which we will see an example very soon). All this is a well developed branch of mathematics, called *deformation theory* and has applications also in other sciences.

Let us consider an example. Physics was over centuries well described and explained by a theory based on *Newton's laws of motion*, formulated by the English scientist Isaac Newton in his *Philosophiae Naturalis Principia Mathematica*, first published in 1687. Nowadays, we refer to this theory as *classical mechanics* in contrast to the theory of *quantum mechanics* which physicists started to develop in the early years of the 20th century driven by the realisation that classical mechanics no longer was adequate to describe phenomena observed in nature.[1]

Astonishingly, it turns out that this fundamental new theory of quantum mechanics can be described as a deformation of classical mechanics. Indeed, it is one of the deepest mathematical results of the 20th century, *Kontsevich's Formality Theorem*, which implies that every classical mechanical system – which we, for example, could imagine to be three particles rotating around each other, held together by gravitation – can be "deformed" into a quantum mechanical one. The theorem is named after the Russian-French mathematician Maxim Kontsevich, born 1964.

To be a little bit more precise: what we mean by deformation of a classical system is the deformation of the multiplication on the set O of *observables* of the system. Observables represent physical quantities which we can measure for a particular physical system. These might be for instance velocity, temperature, or pressure. The outcome of a measurement of an observable is a specific value, which is assigned to the state of the system. The notion of measurement, however, is very different depending on whether we consider classical or quantum mechanics. Accordingly, the mathematical objects which are used to describe observables in a physical theory behave quite differently depending on the context.

What has all this to do with operads? Well, let us take a step back and start from a different direction. You already know the little discs operad $\mathcal{D}$. Now, there is a general process to derive a new operad from a given operad which we will call "taking (co)chains". A detailed explanation how this works would unfortunately take a snapshot on its own, so we will have to be content with knowing that this process exists and that these chains encapsulate essential structural information about mathematical objects.

Anyway, by taking chains of the little discs operad $\mathcal{D}$ (or more specifically speaking, taking so-called *singular chains*) one obtains a new operad. We will denote this operad by $\mathcal{C}$.

On the other hand, one can take the so-called *Hochschild cochains* of the set O of observables and obtains the operad $\mathcal{H}$ of Hochschild cochains of O. This operad $\mathcal{H}$ consists of all functions C which have observables $x_1, \ldots, x_n \in O$ as inputs and one observable as output. Moreover, these functions are *linear* in each variable, meaning

$$C(\ldots, ax + y, \ldots) = aC(\ldots, x, \ldots) + C(\ldots, y, \ldots)$$

[1] More on quantum mechanics and what distinguishes it from classical mechanics you can find for example in the Snapshot 8/2014 *The Kadison-Singer problem* by Alain Valette.

holds for all observables x, y and any number a. It turns out that the operad $\mathcal{H}$ possesses more structure than one guesses at first sight.

In order to formulate this fact, we will make use of the following:

Definition. If $\mathcal{P}$ is an operad, then a *$\mathcal{P}$-algebra* is a set X together with a rule t that assigns to each $P \in \mathcal{P}_n$ some $t(P) \in \mathcal{E}(X)_n$ in such a way that for all $P, Q \in \mathcal{P}$ we have $t(P) \cdot_i t(Q) = t(P \cdot_i Q)$.

When one is dealing with an abstract operad $\mathcal{P}$, it can be helpful to look for such a t because with its help one can represent the elements of $\mathcal{P}$ by concrete functions on a set X. However, sometimes it is the other way round and a $\mathcal{P}$-algebra structure on a set X is what this set really is about: the naked set X is like the jigsaw puzzle with all pieces turned upside down, and the $\mathcal{P}$-algebra structure is like the picture on the back of the pieces.

Indeed, the miracle is:

Theorem. *$\mathcal{H}$ carries the structure of a $\mathcal{C}$-algebra.*

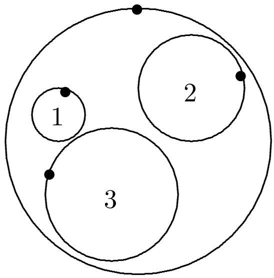
This is *Deligne's Conjecture.* The Belgian mathematician Deligne (born 1944) was first to suppose this statement, however, he left it as a speculation and the extremely hard problem to prove it. Recently, our mathematical community has found different proofs to transform this conjecture into a theorem. Some of the ideas in these proofs are then also used in proofs of Kontsevich's theorem mentioned above.

Indeed, Kontsevich's theorem itself makes highly nontrivial statements about properties of this $\mathcal{C}$-algebra structure and in the end these also imply the existence of deformations of classical systems into quantum systems.

You realise I become more and more sketchy, but all I want is to send you the following message: there was a hard problem in mathematical physics, and extremely clever people had stared at it for decades without succeeding. The solution indeed could first be found when turning the problem upside down, revealing and understanding the hidden structures behind it, namely linking the – at first sight completely unrelated – Hochschild cochains $\mathcal{H}$ to the action of the singular chains of the little discs operad $\mathcal{D}$.

3.3 Outlook

We arrive at a different field of research by a variation of the little discs operad $\mathcal{D}$. For this, we modify the little discs a bit by putting a marker on the boundaries, with the ambient disc marked at the top:



Now, also the "product algorithm" for $P \cdot_i Q$ has to be adjusted: when gluing Q into P we have to make sure to rotate the disc Q first, so that the marker on the boundary of Q matches the marker on the i-th disc in P. We call the resulting new operad the *marked little discs operad*.

As has become clear not so long ago, algebras over the singular chains of that marked little discs operad define so-called *Batalin-Vilkovisky algebras*. Even if we at this point cannot go into what exactly these algebras are, it is interesting to know that they form a mathematical structure which itself was first discovered on so-called *ghost fields* of quantum field theories, which play an important role in modern physics.

I myself stumbled across Batalin-Vilkovisky algebras in my own research in yet another, completely different field called *Hopf algebras*. A goal of research within these topics is also to bring physicists, geometers and algebraists together and understand at least some of the hidden connections between these fields.

This text hopefully gave you some idea what operads are and why we study them. The references below contain more details and further references, but they are mostly written for research mathematicians, so please do not hesitate to contact me if you have any questions!

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Mathematical subjects
Algebra and Number Theory, Geometry and Topology

Connections to other fields
Computer Science, Physics

DOI
10.14760/SNAP-2015-017-EN

Snapshots of modern mathematics from Oberwolfach are written by participants in the scientific program of the Mathematisches Forschungsinstitut Oberwolfach (MFO). The snapshot project is designed to promote the understanding and appreciation of modern mathematics and mathematical research in the general public worldwide. It started as part of the project "Oberwolfach meets IMAGINARY" in 2013 with a grant by the Klaus Tschira Foundation. The project has also been supported by the Oberwolfach Foundation and the MFO. All snapshots can be found on www.imaginary.org/snapshots and on www.mfo.de/snapshots.

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