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Operator theory and the singular value decomposition

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This is a snapshot about operator theory and one of its fundamental tools: the singular value decomposition (SVD). The SVD breaks up linear transformations into simpler mappings, thus unveiling their geometric properties. This tool has become important in many areas of applied mathematics for its ability to organize information. We discuss the SVD in the concrete situation of linear transformations of the plane (such as rotations, reflections, etc.).

1 What is operator theory?

Operator theory is the study of transformations of infinite-dimensional spaces and is a natural framework for analyzing a variety of problems inspired by physics. One such problem is the *Dirichlet problem*: if you heat up the edge of a metal plate, what will be the temperature at a given point in the center? This is an *infinite-dimensional problem* because the temperature distribution on the edge of the plate has an infinite number of parameters. This is not obvious, but you could imagine that you have the freedom to adjust the temperature at any given point on the edge of the plate without changing the temperature at other points too much and so there are an infinite number of parameters to play with.

In addition to the Dirichlet problem and related problems, operator theory most notably can be used as a mathematical framework for quantum mechanics.[1]

Discussing what infinite-dimensional spaces are is beyond the scope of this snapshot, but it is still possible to give an idea behind at least one result from operator theory: the singular value decomposition (SVD). This is an old but powerful result that allows one to uncover the precise geometric behavior of a linear transformation. The SVD has become popular and even famous [2] in recent years because it can be used as a "blackbox algorithm". Plug your favorite data into an array of numbers or a matrix, find the singular value decomposition, and voilà! You may just uncover hidden patterns and connections in your data with little human involvement.

2 Linear transformations

The singular value decomposition is about breaking up so-called *linear transformations* into simpler pieces that give a clear idea of their geometric properties. To describe it, we will look at the simplest interesting situation of linear transformations in two dimensions.

Some simple examples of linear transformations of the plane are rotations, reflections, stretchings/shrinkings, and projections; see Figure 1, which shows the unit square, the result of a rotation, and the result of a stretching in the vertical direction and a shrinking in the horizontal direction.

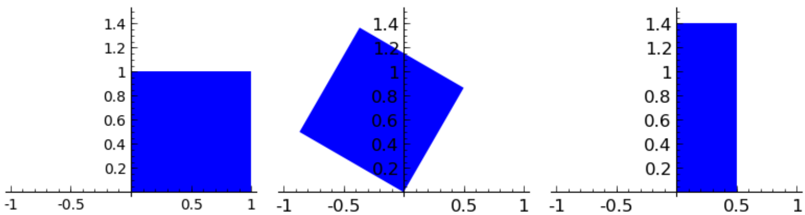

Figure 1: The transformed unit square

Essentially, a linear transformation sends the point $(0, 0)$ to $(0, 0)$ and sends parallelograms to parallelograms. In algebraic terms, a linear transformation

[1] A brief introduction to quantum mechanics and more on the subject can be found in the following snapshot: Alain Valette, *The Kadison-Singer problem*, Snapshots of modern mathematics (2014), no. 8, 1–5.

takes any point (x, y) in the plane and transforms it to $(ax+by, cx+dy)$, where a, b, c, d are constants:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

The *shearing transformation* in Figure 2 is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ x + y \end{pmatrix}.$$

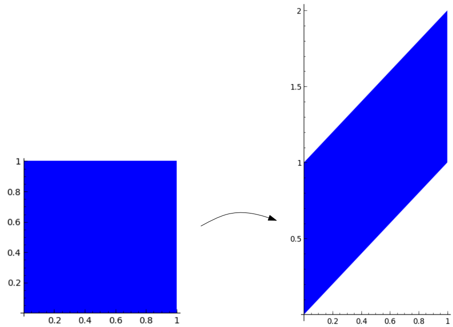

Figure 2: Shearing the unit square

This transformation seems quite simple, but when we examine what it does to the unit circle instead of the unit square, its behavior looks complicated as in Figure 3.

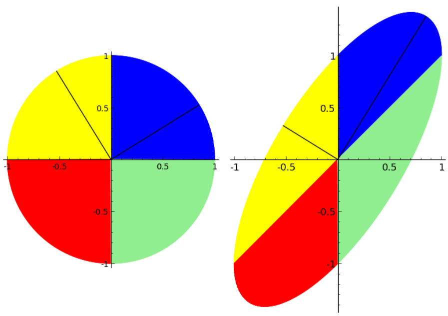

Figure 3: Shearing the unit circle

3 The singular value decomposition

Is it possible to break up our shearing transformation into simpler transformations like rotations, reflections, and stretchings/shrinkings? The singular value decomposition tells us exactly how to do this and it tells us that *every linear transformation can be broken down into a rotation/reflection followed by simple stretching/shrinking followed by a rotation.*

We will illustrate this with the shearing transformation. In Figure 3, the black lines on the circle indicate the lines through $(0, 0)$ which get stretched the most and the lines stretched the least by the transformation as shown on the ellipse. If we follow what happens to these lines we can give a nice description of the shear transformation. Figure 4 shows how this is done. First, rotate the circle to put the black lines on the axes. Next, stretch in the horizontal and vertical directions by the appropriate amount. The factors that you stretch by are called the *singular values* of the transformation, and this is where the SVD gets its name. Finally, rotate the ellipse into place.

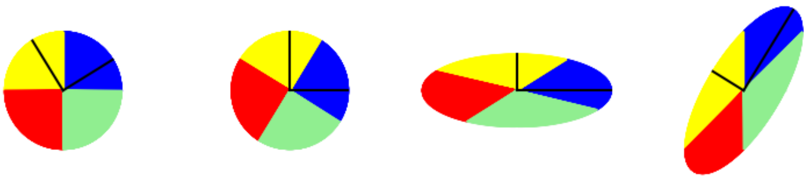

Figure 4: Singular value decomposition

The SVD is extremely general. Linear transformations in three dimensions turn spheres into ellipsoids (see Figure 5), and a similar process can be used to break it into rotation/reflection followed by stretching followed by rotation.

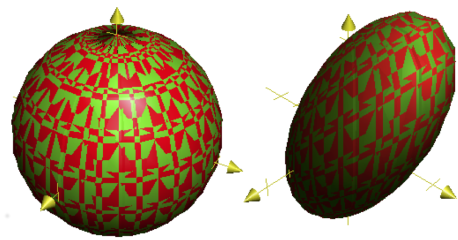

Figure 5: Three-dimensional linear transformation

The SVD in fact works in any number of dimensions and even in infinite dimensions, although in infinite dimensions you must be a little more careful about what you mean by a rotation or a stretching because of the following fact: in infinite dimensions rotations are replaced by *isometries*, that is by linear transformations that preserve length. Unfortunately, isometries in infinite dimensions can transform the entire space onto a proper subset of itself![2] An important example of an infinite-dimensional space is the collection of all infinite sequences of real numbers

$$(x_1, x_2, x_3, \dots)$$

which also satisfy the condition that $x_1^2 + x_2^2 + x_3^2 + \ldots < \infty$.[3] The *length* of such a sequence is defined as the number $\sqrt{x_1^2 + x_2^2 + x_3^2 + \ldots}$.

An example of an isometry on this space is the *shift operator*

$$(x_1, x_2, x_3 \dots) \mapsto (0, x_1, x_2, x_3, \dots).$$

The shift is an isometry, since the length of the sequence $(0, x_1, x_2, x_3, \ldots)$ is $0^2 + x_1^2 + x_2^2 + x_3^2 + \ldots$, that is, equal to the length of the sequence $(x_1, x_2, x_3 \ldots)$. You will notice that the shift operator transforms the whole space onto the proper subset of itself consisting of sequences with first entry 0. So, isometries in infinite dimensions do more than just rearrange the space, they actually can push the space away while paradoxically staying inside itself. Even though isometries are much more complicated in infinite dimensions, there is a rich theory behind them – there is even a rich theory behind the simple looking shift operator above – so the SVD remains a useful tool even in this setting.

4 Further reading

We have just scratched the surface of the SVD. A longer article at a similar level discussing the singular value decomposition which we recommend is [1]. Wikipedia has a nice discussion of the SVD [3].

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[2] Meaning that if you apply the transformation to the whole space, what you get might not be the whole space, but possibly less.

[3] This means that one requires that there exist a real number (symbolically denoted by $x_1^2 + x_2^2 + x_3^2 + \ldots$) such that the finite sums $x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2$ come arbitrarily close to it, but never exceed it. Any sequence that has only finitely many entries different from zero fulfils it, since after finitely many additions you only add zero to the sum. But there are also many other sequences $(x_1, x_2, x_3, \ldots)$ that fulfil $x_1^2 + x_2^2 + x_3^2 + \ldots < \infty$.

References

[1] David Austin, *We recommend a singular value decomposition*, http://www.ams.org/samplings/feature-column/fcarc-svd, 2009, Online; accessed 14-July-2014.

[2] Clive Thompson, *If you liked this, you're sure to love that*, http://www.nytimes.com/2008/11/23/magazine/23Netflix-t.html, November 23, 2008, p. MM74.

[3] Wikipedia, *Singular value decomposition*, https://en.wikipedia.org/wiki/Singular_value_decomposition, Online; accessed 14-July-2014.

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Mathematical subjects
Analysis

Connections to other fields
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