

# LUNAR PARALLAX MEASUREMENT – DIRECTIONS AND THEORY FOR LOWER AND UPPER SECONDARY SCHOOL STUDENTS

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**Abstract:** Measuring the distance to the Moon is not a particularly difficult task for high school students and amateur astronomers. This article explains the required measurements and outlines the details of the required calculations. The project is suitable for lower and upper secondary school students. To increase the reliability and accuracy of the experiment the distance between the observation points should be maximised, ideally situated on different continents. This will also promote co-operation between inter-continental amateur astronomers which is a secondary aim of the project.

## Introduction

Parallax is a term used to describe a phenomenon where, due to the different viewing positions of two observers, there is a difference in the apparent position of an object with respect to its background. The value of this apparent displacement also depends on the distance from the object to the observer. Based on the apparent displacement of the observed object and some simple mathematics, it is possible to determine the distance to the object. In addition to manipulation of certain mathematical formulae, specific distances are also needed; the measurement of these will be explained as required. This method is very useful for inaccessible objects that cannot be physically measured. This is one of the reasons why the measurement of parallax was, and still is, widely used in astronomy.

This traditional old method is now even more straightforward due to technological advances and ease of access to digital photography and printers. The method will be introduced using distances that can be physically measured and then applied to measuring the distance to faraway objects. This method will then be applied to measuring the distance to the Moon. The calculation has been modified to ensure that it is at an appropriate mathematical level for lower and upper secondary school students.

## Parallax

Three chairs in the room were positioned facing a window as shown in Figure 1. A neighbouring house and its chimney were visible from the window. Is it possible to calculate the chimney distance from the observer?



Figure 1. The chairs (A, B and C) facing the window.

The chimney was initially photographed as seen from chair A and then as seen from chair B. The images were compared on paper (or on screen) and it was observed that the distance between the chimney and a distant bell tower (situated on a hill behind the chimney) appears larger in the second image. This is shown by the red double arrow. The second image was taken from chair B. Both of these images are shown in Figure 2.



Figure 2. Picture taken from chair A (left) and chair B (right).

In order to calculate the distance to the chimney, it was first necessary to create a reference image and measure the displacement of the camera. These would be used as intermediary standards. This was carried out by moving the camera from chair A to chair B. Initially a reference image of an object situated approximately 30 m away was required. The focal length of the camera lens was kept constant as  $f = 80$  mm. This was the setting used for all images in this article. A wooden plank of length 1.205 m was then placed 28.0 m from the camera and photographed. This image was then displayed on a screen (or paper) at the same magnification (same size) as the chimneys in the previous image. The plank's image was only 74 mm long. The black cloths were placed at the ends of the plank giving better visibility (Figure 3).



Figure 3. A white plank, positioned at a distance of 28.0 m.

The reference image (photograph of the plank) and both images of »chimney with background« were then displayed using the same magnification. The length of the plank in the image was then measured and found to be 74 mm. If the chairs were exactly 1.205 m apart and the apparent shift of the chimney was exactly 74 mm, we would already have our result: in this case, the distance from the chairs to the chimney would be 28 m. The equation required to evaluate this is relatively straightforward. The distance to the measured object ( $D$ ) equals the distance to the reference object; let's call this distance  $plank\_dist$ , distance to the plank:

$$D = plank\_dist$$

How must the equation be modified so that it would be valid when the apparent shift of the chimney (in the picture) is not the same as the plank length (in the picture)? We know from experience that the apparent shift is smaller for more distant objects. (To test this: hold a finger about 30 cm in front of your eyes, you will observe a shift of the background objects for a distance  $x$ . When your finger is then moved 60 cm away from your eyes, the apparent shift, when you look at the finger with the left and the right eye separately, will be only  $x/2$ .) A smaller apparent shift of the chimney with regards to the plank length in the picture means that the chimney is farther away than the plank. So we will add into the equation the plank length in the image ( $plank\_length\_image$ ) and apparent shift of the chimney ( $app\_shift$ ). Half a shift would mean double the distance, a quarter - quadruple etc... This results in the modified formula:

$$D = plank\_dist \cdot \frac{plank\_length\_image}{app\_shift}$$

This is a more general equation however the possibility that the plank could be a different length than the distance between the two chairs must be considered. The question now is how to include the plank length ( $plank\_length$ ) and the distance between the chairs ( $chair\_dist$ ). If the chairs were twice as far apart as the length of the plank and a very small apparent shift of the chimney with regards to the background was observed, we would expect the chimney to be very far away. On the other hand, if we observed a large apparent shift of the chimney with the chairs close together, this would mean that the chimney is quite close. The distance to the measured object is apparently proportional to the distance between the observation points and inversely proportional to the »plank length« (a reference object, which describes the grazing angle of our snapshot). For this reason another term is added:

$$D = plank\_dist \cdot \frac{plank\_length\_image}{app\_shift} \cdot \frac{chair\_dist}{plank\_length}$$

Our simple thought experiment has generated empirically a general distance equation. This can now be used to calculate the distance to our chosen object. When considering parallax, an angle is generally used and must also be measured. In this case, the equation disguises the angle in the form of a ratio.

Generality aside, let us test our equation. The distance between the chairs was 0.50 m and the apparent shift of the chimney in the image was 27 mm. This leads to:

$$D = 28 \text{ m} \cdot \frac{74 \text{ mm}}{27 \text{ mm}} \cdot \frac{0.50 \text{ m}}{1.205 \text{ m}} \approx 32 \text{ m}$$

The result is very close to the distance that we get with direct measurement. Of course the critical reader will be tempted to complain about the neatness of the equation, meters and millimeters should not really be mixed in the same equation. However, since the millimeters cancel out, the method is mathematically and dimensionally acceptable.

If the pictures were taken from chairs C and B instead, the image obtained from chair C would be virtually identical to that from chair A. However if the distance between chairs C and A is large, the value of the chimney's displacement would be significantly greater than the distance calculated using the measurements from chairs A and B. Chairs C and B, can also be used, but in this case the distance between the observers is not the distance between the two chairs, but rather the "perpendicular\_chair\_distance". (see Fig.4).



Figure 4. Instead of the distance between B and C (dashed line) we need to take the »perpendicular distance«, which is marked by the solid line. The solid line is perpendicular to C-object and also to B-object line, as these two lines are almost parallel.

$$D = plank\_dist \cdot \frac{plank\_length\_image}{app\_shift} \cdot \frac{perpen\_chair\_dist}{plank\_length}$$

### Bright, bright Moon

There are few problems when photographing the Moon, although a sturdy tripod is required. The main problem with such measurements is to get both the Moon and the background stars in the exposure of the photograph. To obtain a good photograph of the Moon we require an exposure of about one hundredth of a second, while the background stars need several seconds exposure. During such a long exposure, the Moon will be much too bright, have fuzzy edges and the result will be a strange very bright unfocused spot. There are at least two ways in which this problem can be resolved.

The first way can be achieved with minimal equipment. The exposure time is set to 6 seconds. The camera focus is set to infinity, with focal length the same as used for the reference image. An object to shadow the Moon needs to be set up in front of the camera. After approximately 5 seconds the object is quickly pulled aside and the other hand used to obscure the lens. With this we have given the Moon just a fraction of second to send the light into the camera, while the neighbouring stars



were given more time. There is no need to worry about the last second or two, when the hand or a cardboard obscures the camera lens (see Figure 5).



Figure 5. An object is used to shadow the lens, but allows the light from the stars to reach the camera.

With more sophisticated equipment, it is possible to use the following method as suggested by Mr. Oleg Toumilovich, South African astrophotographer: “To have the stars and the Moon in one shot a filter of a suitable density is attached on a 2 mm steel (or even better copper) rod, directly onto the camera mount. This ensures that the disc stays in the centre of the FOV irrespective of the camera orientation. The camera is moved until the disc fully covers the moon.” This is shown in Figure 6.



Figure 6. Advanced option to achieve a balanced exposure of stars and Moon.

### **Participation (and cooperation)**

When taking pictures of the neighbour's chimney it was possible to easily move from one chair to another. However when photographing the Moon, the photographer would need to move a few thousand kilometres between pictures to get an image with a usable apparent displacement due to the great distances involved. Another problem is the movement of the Moon during the photographer's travels resulting in different stars forming the image background. To apply the parallax measurement to determine the distance to the Moon, the image needs to be taken at the

same time from two different observing points. In practice, this means that one needs to find a willing colleague who will take a picture of the Moon on the other side of the world. South Africa and Europe can be good complementary choices. Finding a volunteer is not an easy task, but the rewards are definitely worth it. Six months ago the first volunteer was Mr. Auke Slotegraaf, this time the weather conditions were favourable for Mr. Oleg Toumilovich; both observers are from South Africa and are members of the Astronomical Society of Southern Africa.

## Images

The photographs can be taken in a few minutes, although the preparation and coordination is far more time consuming. The weather at both places needs to be taken into account and the available time of the cooperating partners. Oleg, in South Africa (SA) used a balancing filter to take his image, while the author chose to use the method of physically shadowing the Moon – this approach requires some skills but less equipment (no filter is needed). In order to determine the direction to the Moon, a perpendicularly placed rod was used and the shadow it cast was sufficient for this measurement. The rod was 88 cm high; the shadow was 125 cm to the west and 191 cm to the north. This data will be used later.

At first glance the image taken using the method shown in Figure 5 seems to be unsuitable (Figure 7a). The same could be said for the image taken using Oleg's advanced method (Figure 7b). However, both images were perfect after some computer manipulations to enhance the images were carried out.

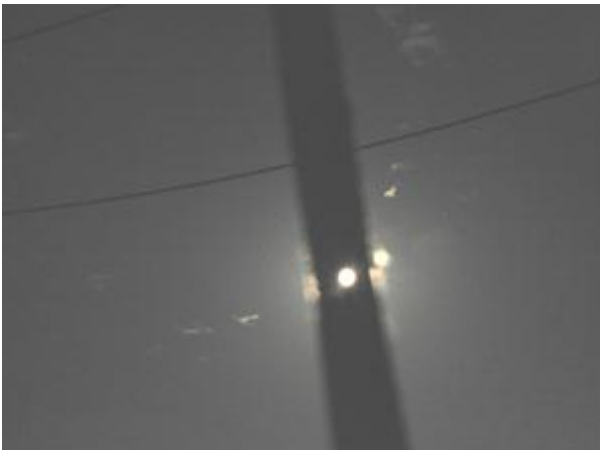


Figure 7a. Image of the stars and the Moon taken by the method as shown in Figure 5.



Figure 7b. Image taken by the method as shown in Figure 6. Photo: O. Toumilovitch.

To obtain a picture where both Moon and stars can be clearly seen, some image image-processing software was used. One of the methods was to first use brightness, contrasts and midtone adjustment functions to make the stars clearly visible (the Moon will be much too bright and without sharp edges in this phase). The next step was to open the modified file in MS Paint® or a similar program. It is now possible to locate the position of the brightest stars to an accuracy of a pixel. The coordinate position of each star was recorded. Next, a sharp image of the Moon was created from the original image; however the stars were invisible due to the darker image. This file was then saved and reopened in MS Paint®. Using the stars' coordinates their positions were drawn at the correct position, as a white square of nine pixels for each star. At a magnification of 800%, this is a relatively straightforward task. The negative of the image was used in order that printer ink usage was minimised.

The author's image was printed on a piece of paper and attached with Scotch tape to the window. The second image, taken by Oleg in SA, was enlarged so that the star images coincided. This was then superimposed on the author's image, while the Moon was displaced by a bit more than one diameter as shown in Figure 8. To ensure calculation accuracy, special care needed was taken to ensure that the image from Ljubljana was at same magnification and focal length settings as the reference image. Certainly the focal length of the South Africa camera could be different. What matters is the magnification of their image.

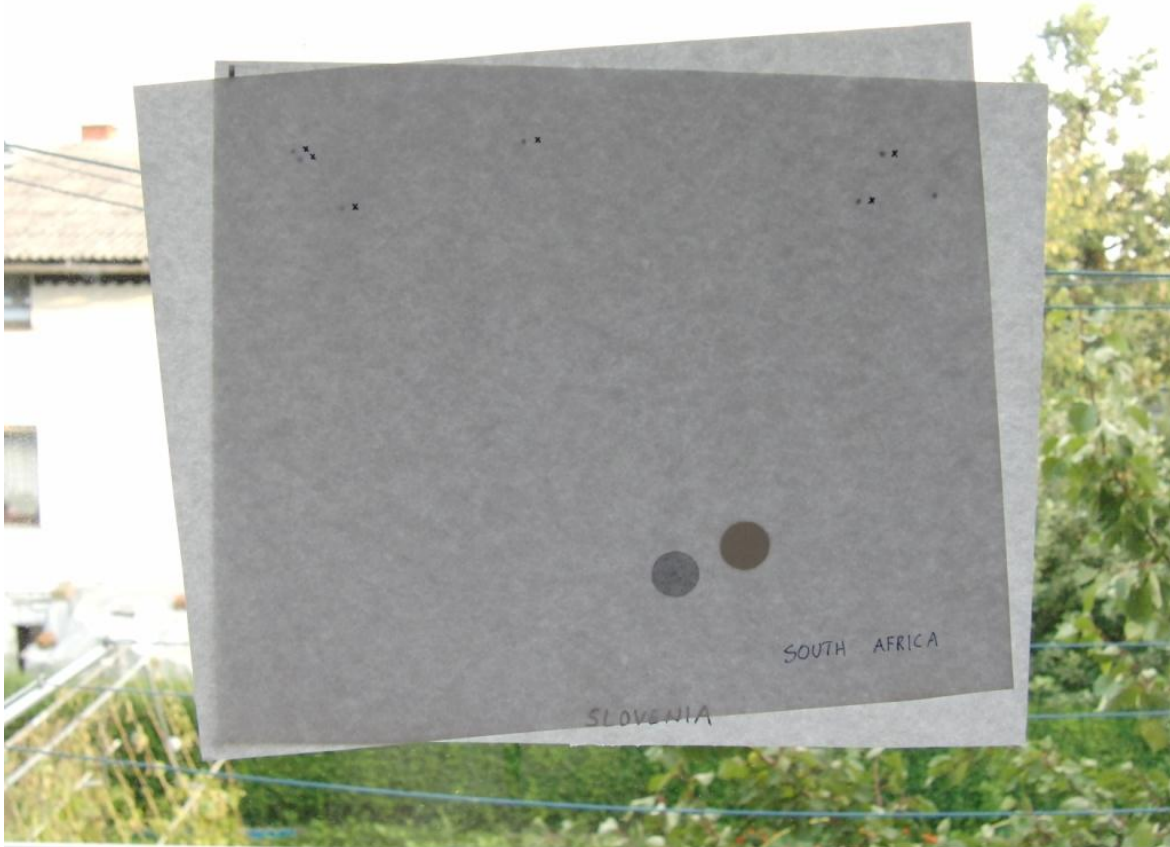


Figure 8. The image from observing point Ljubljana, Slovenia printed on paper on the window. The second image from Johannesburg-South Africa covers it. It is intentionally displaced slightly to observe the coincidence of the stars. Slovenian stars are left dots, while S. African stars are right. However the South African Moon is on the left while the Slovenian Moon is on the right. The easier part of the task was to measure the apparent displacement of the centre of the Moon with regards to the background stars (in this case 29 mm with no star displacement). It was far more difficult to determine the distance between the two observers – is it the case of chairs A and B or chairs B and C?

## Finding the distance to the Moon

### 1 A (mathematically) simpler method

A basic knowledge of trigonometry is required in order that an accurate calculation of the distance to the moon can be carried out. However it is also possible to achieve this by a simpler, but less accurate method. This is particularly suitable for lower secondary school students and amateur astronomers who prefer a less mathematical approach. The alternative more mathematical (and more accurate) method for calculating the distance between the observers will also be described.

The simpler method is based on a measurement of the distance using a globe as shown in Figure 9. A globe of diameter about 30 cm is suitable for this method. The globe should be removed from its stand and placed in a bowl so that it can rotate freely. The globe was oriented so that Ljubljana was at the very top (as shown below). Two pencils and a rod (used as a pointer) were clamped on a retort (clamp) stand and positioned near the globe. The rod was positioned over the line of longitude from Ljubljana towards south. One pencil was positioned to touch Johannesburg. The tip of the second pencil was situated on a vector line from Ljubljana pointing towards the Moon. The pencil position was determined by going 191 mm towards south on the rod, then 88 mm directly upwards and finally 125 mm east.

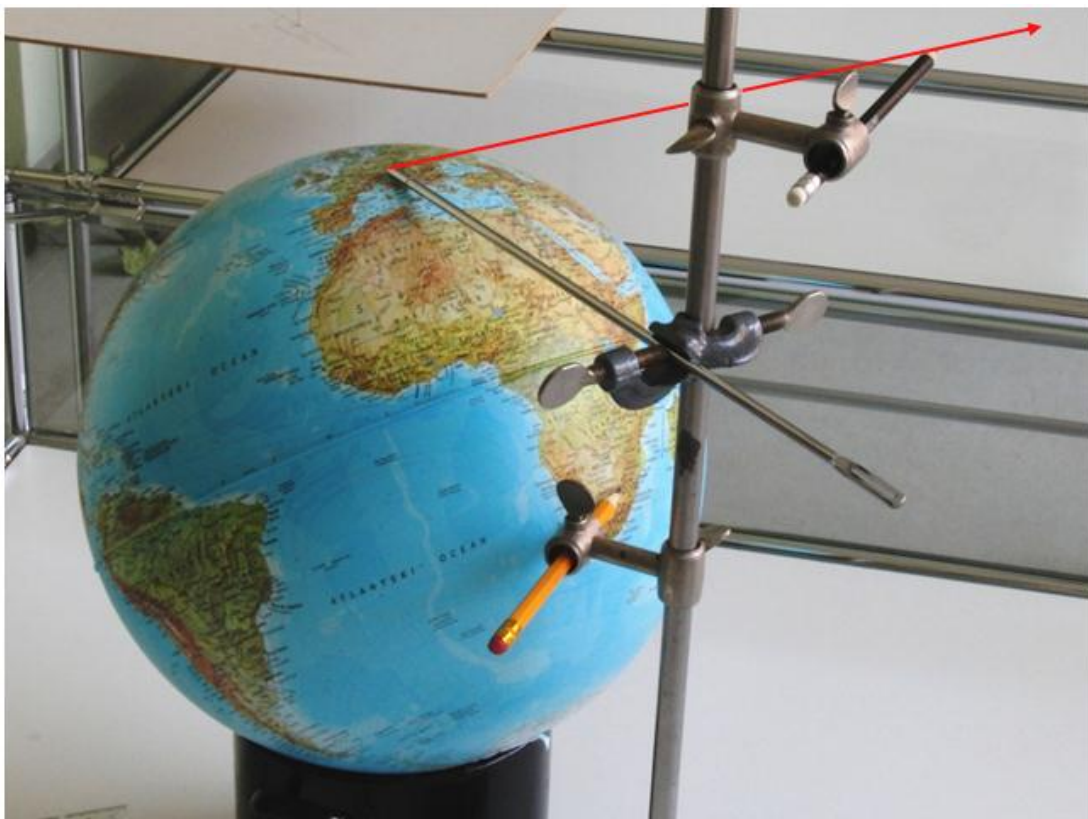


Figure 9. The positions of the rod and two pencils are determined by the position of Ljubljana, Johannesburg and the direction towards the Moon (as shown by the red arrow).



The two pencils and the rod remain set up as described above and the globe is removed and replaced with a hard piece of cardboard so that it touches the points of the pencils. All three intersection points are marked as shown in Figure 10. This is a representation the position of chairs B and C.

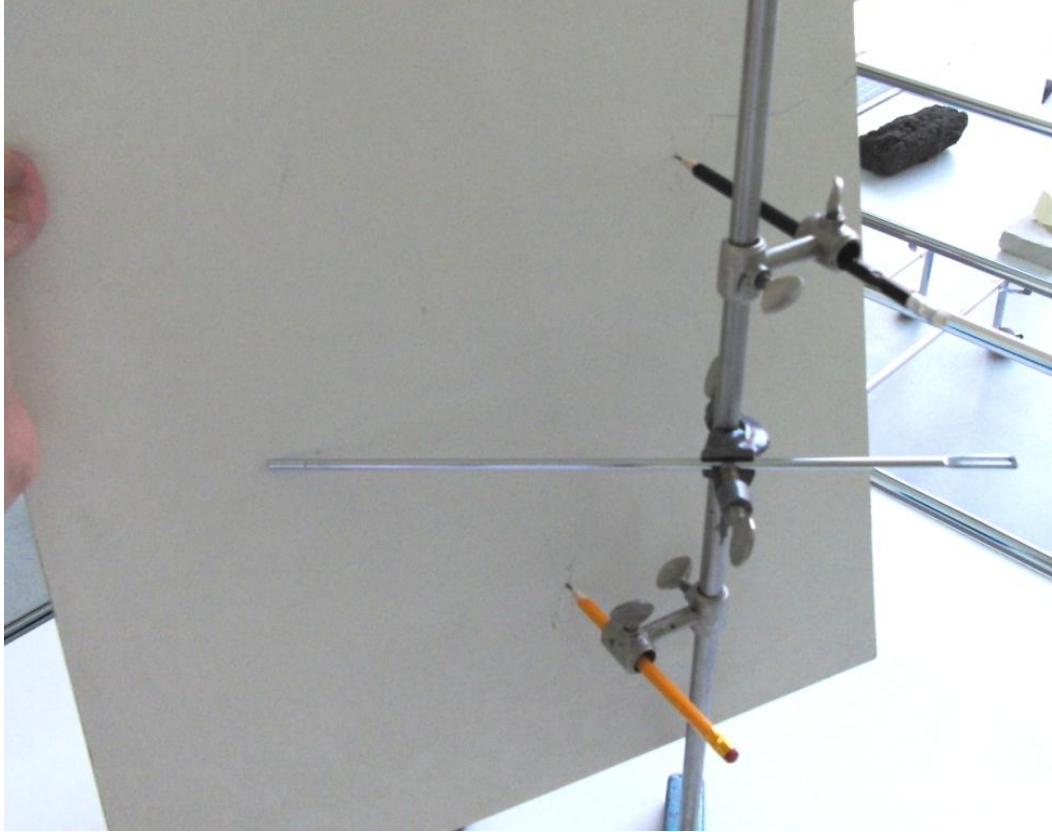


Figure 10. A simple method to determine the distance between observation points: the three points are marked on cardboard.

The three points define a triangle which was then drawn on the card. A line orthogonal to the Ljubljana - Moon line was drawn passing through Johannesburg as shown in Figure 11. The distance between Ljubljana and Johannesburg on the card was  $15.6 \text{ cm} \pm 0.5 \text{ cm}$ . As long as the globe scale is known then the actual distance between Ljubljana and Johannesburg can be calculated. It was found to be  $6625 \text{ km} \pm 200 \text{ km}$ .

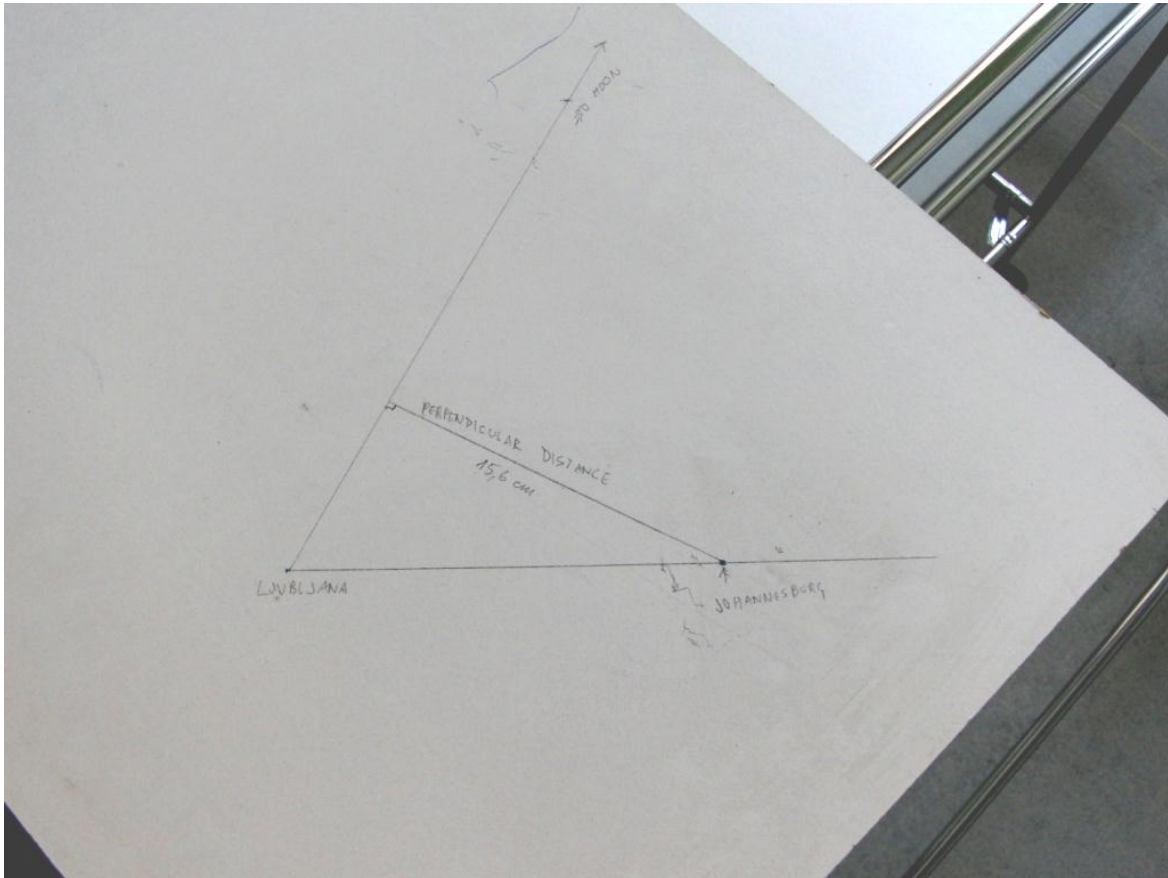


Figure 11. Johannesburg corresponds to “chair B” while Ljubljana obviously corresponds to “chair C”. Therefore “perpendicular distance” must be used in the parallax calculation (See also Figure 4).

The distance to the Moon is then:

$$D = plank\_dist \cdot \frac{plank\_length\_image}{app\_shift} \cdot \frac{perpen\_chair\_dist}{plank\_length}$$

Using measured data:

$$D = 28 \text{ m} \cdot \frac{74 \text{ mm}}{29 \text{ mm}} \cdot \frac{6625 \text{ km}}{1.205 \text{ m}} \approx 393 \text{ 000 km}$$

A measurement uncertainty of about 4% was estimated. Therefore the Moon distance is between 377 and 409 thousand km.

## 2 Using trigonometry

The model of the Earth was positioned in a three-dimensional coordinate system, as is shown in Figure 12. The line of longitude passing through Ljubljana is in the  $zx$  plane. The coordinates of Ljubljana are  $14.7^\circ$  E and  $46.2^\circ$  N.

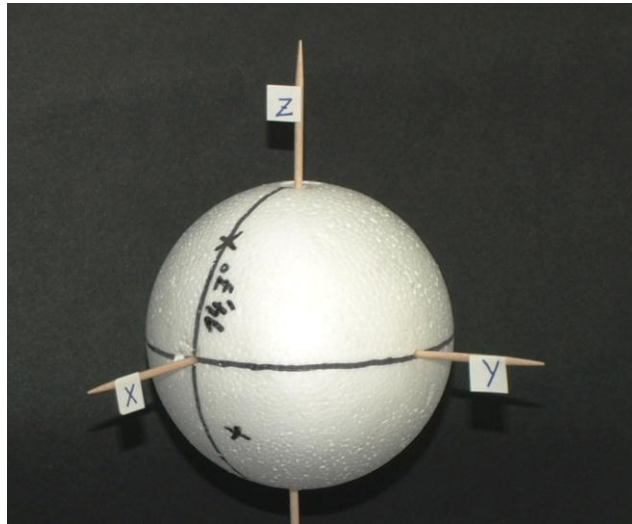


Figure 12. A styrofoam ball served as a model of the planet Earth. The author's observing point was the upper cross, the observing point in Johannesburg is marked by the lower cross.

The radius of the Earth is 6 370 km and the coordinates of Ljubljana (in the Cartesian coordinate system used above) are (4409, 0, 4598) units of km. Oleg's position in Johannesburg, South Africa,  $26^\circ$  S and  $28^\circ$  E, has coordinates (5572, 1317, -2792). Using Pythagoras, the *distance between the observers* was calculated as:

$$d_{obs} = 7\,596 \text{ km}$$

It also needs to be established if this corresponds to the case of "chairs B and C". The angle between line segments Ljubljana – Johannesburg and Ljubljana – Moon must be calculated. To do so, the coordinates need to be transformed twice. To do this Ljubljana is made the origin of the coordinate system. In this case it will be easy to write the vector pointing from Ljubljana towards the Moon. The first step is a rotation around  $y$ -axis, which will position Ljubljana on the top of the Earth model. The angle is  $\varphi = 43.8^\circ$  and only  $x$  and  $z$  coordinates change. The new coordinates will be:

$$z_n = z \cos\varphi + x \sin\varphi$$

and

$$x_n = -z \sin\varphi + x \cos\varphi$$

Ljubljana is now positioned at (0, 0, 6370) and Johannesburg is at (5954, 1317, 1841). The second transformation will be a parallel translation (6370) in the direction of the  $z$ -axis. The coordinates of the first observing point will become the origin of the coordinate system (0, 0, 0). The same

transformation means that Johannesburg is now at (5954, 1317, -4529). The longitude line  $14.7^{\circ}$  is still in the  $xz$  plane.

Because Ljubljana is now at the origin of the coordinate system, the vector towards our South African colleague is the same as the coordinates. The other vector, obtained from the rod that was illuminated by the Moon, in this coordinate system points from Ljubljana to the Moon direction: (191, 125, 88). Now all that still remains to be done is to calculate the angle between these two vectors and use it to determine the distance between “chairs A and B”.

The angle between these vectors is given by:

$$\cos\varphi = \frac{\vec{m}\vec{n}}{mn},$$

The first vector is  $\mathbf{m} = (5954, 1317, -4529)$  and the second  $\mathbf{n} = (191, 125, 88)$ , and so  $\varphi = 60.9^{\circ}$ . The distance that needs to be inserted into the parallax equation is

$$d_p = d \sin\varphi,$$

which gives 6 638 km (so called parallel distance). Finally, we can now calculate the distance to the Moon:

$$D = \text{plank\_dist} \cdot \frac{\text{plank\_length\_image}}{\text{app\_shift}} \cdot \frac{d_p}{\text{plank\_length}}$$

and

$$D = 28 \text{ m} \cdot \frac{74 \text{ mm}}{29 \text{ mm}} \cdot \frac{6638 \text{ km}}{1.205 \text{ m}} \approx 394 \text{ 000 km}$$

We estimate that the distance between the Earth and the Moon is  $(3.94 \pm 0.10) \cdot 10^5$  km. The estimated error is  $< 3\%$ . The Stellarium program gives  $3.90 \times 10^5$  km at the experimental time.

## Conclusion

The procedure has been described in such a way to make the experiment accessible to lower secondary school students. Another important point to note is that no advanced apparatus was required and students could carry out the experiment using equipment they already have at home. The calculation can be performed in two ways giving the option of a mathematical solution for the more advanced students.

We sincerely hope the suggested procedure will help teachers from different continents to give their students a real world application of mathematics and to understand its power as a scientific tool. The perfection and appeal of the virtual world (using programs like Stellarium), is gaining momentum, and indeed the field of astronomy is nowadays so comfortable using such virtual tools that they sometimes forget about the possibility for real-world exploration and observation. However, it is hoped that the excitement of actually measuring the parallax will motivate the students to pursue further observations and measurements of the night sky.

**Acknowledgement:** I would like to thank the following people who helped during the experimentation and write up of this project. An important part of the measurement was the African continent Moon photography. This was carried out by Mr. Oleg Toumilovich from South Africa, and I am grateful for his contribution. I would also like to thank the two translators, Dr. Barbara Šetina Batič and Mr. Andrew McPhee, who successfully expressed my ideas in English from the original Slovenian text.