Calculation and Imagination in the Geometric Work of Johannes Kepler

Uli Gaenshirt, Nuremberg, Germany

1. Introduction

In the 16th century Europe the geometry of regular polyhedra was a matter of particular interest [1]. The drawings of the Renaissance artist Leonardo da Vinci (1452 – 1519), published in 1509 in Luca Pacioli's book *De Divina Proportione* [2], had far-ranging influence about art and philosophy. In 1568, the goldsmith Wenzel Jamnitzer (1508 – 1585) from German Nuremberg presented in his work *Perspectiva Corporum Regularium* [3] phantastic, polyhedral variations, derivated from perspective delineations of the fife Platonic solids.

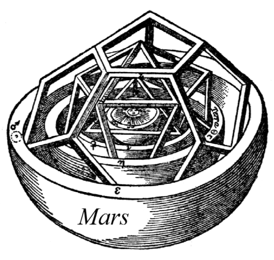

Figure 1. Inner section from *Mysterium Cosmographicum*

The faces of the Platonic solids are regular polygons. Each solid has congruent faces and identical corners. In his first book *Mysterium Cosmographicum* [4], published in 1596, the German astronomer and mathematician Johannes Kepler (1571 – 1630) postulated a direct correlation between the planetary orbits and the circumscribed and inscribed spheres of the Platonic solids. Figure 1 shows the inner part of his cosmological model, with the sun in the center of the four concentric spheres of Mercury, Venus, Earth, and Mars, and with the three intermediate Platonic solids, octahedron, icosahedron, and dodecahedron.

In his main work *Harmonices Mundi* [5], published in 1619, Kepler revised his early ideas, due to the fact that both his cosmological measurement and his polyhedral calculations didn't exactly confirm this model. He remarked that it probably looks too simple to be the work of the artificer of the universe.

2. Measures and ratios of the Platonic solids, tetrahedron, cube, and octahedron

In book II of his work *Harmonices Mundi* [5] Kepler determined the regular polyhedra systematicly from a strong mathematical point of view. In book V he investigated the relationships of the Platonic solids among themselves. These correlations are important to establish a complex understanding of the regular polyhedra.

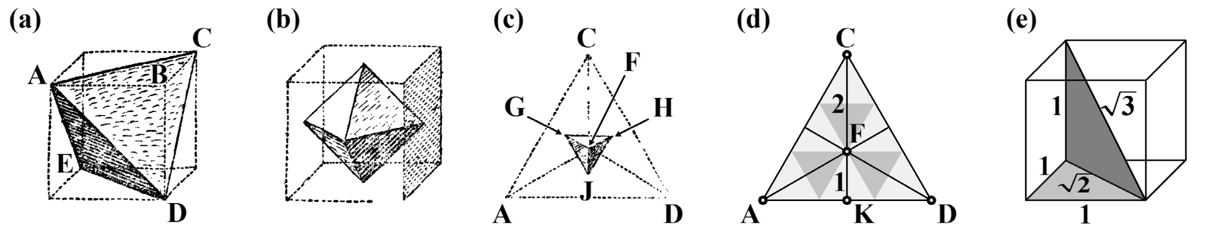

Figure 2. (a – c) Woodcuts from *Harmonicis Lib. V, Cap. I.* (a) Tetrahedron **ACDE** inscribed in a cube. (b) Octahedron inscribed in the dual cube. (c) Tetrahedron **FGHJ** inscribed in the self-dual tetrahedron **ACDE** (the corner **E** is hidden behind the inscribed tetrahedron **FGHJ**). (d) Equilateral triangular face **ACD** of the tetrahedron **ACDE.** The small highlighted equilateral triangles clarify the **3:1** ratio of **CK** and **FK**. (e) Unit cube with face diagonal **√2** and space diagonal **√3**.

Kepler was aware that the volume of the regular tetrahedron **ACDE** in figure 2a is one third of the volume of the cube, spanned by **BA**, **BC**, and **BD**, because the irregular tetrahedron **ACDB** has half the volume of **ACDE** (The ratio of their volumes depends from the ratio of their heights about the face **ACD**, which is as **1** to **2**).

With regard to figure 2b he comments, that the octahedron has the sixth volume of the circumscribed dual cube, because the octahedron has half the volume of the tetrahedron **ACDE** (Please note that an octahedron is a tetrahedron less four small copies from itself with half the edge length: $\mathbf{V_{oct} = V_{tet} - 4\ (V_{tet} / 8) = 0.5\ V_{tet}}$).

Relating to the tetrahedron **ACDE** in figure 2c he specified that the ratio of the radii of the circumscribed sphere to that inscribed in it is as **100000** to **33333** (Compare figure 2d: The geometry inserted into the face **ACD** facilitates to understand, that the ratio of the heights of the tetrahedra **FGHJ** and **ACDE** is as **1** to **3**).

The Pythagorean theorem applied to the triangles in figure 2e reads: $\mathbf{1^2 + 1^2 = (\sqrt{2})^2}$ and $\mathbf{1^2 + (\sqrt{2})^2 = (\sqrt{3})^2}$. That is why the ratio of the radii of the circumscribed sphere of the cube to that inscribed in it is as **√3** to **1**.

Kepler specified this ratio as **100000** to <u>**57735**</u>. The calculator accordingly confirms: $\mathbf{1/\sqrt{3} = 0.\underline{57735}0269\ldots}$.

3. The reconstructed calculation of the circumscribed and inscribed spheres of the dodecahedron

In figure 3a, Kepler demonstrates how to subdivide a dodecahedron into a central cube and six roof-shaped trapezohedrons. One of them, with the originally marked corners, **A**, **B**, **C**, **D**, and **E**, he seperately drawed in.

The denotations, **T**, **P**, **G**, **F**, **Q** and **J**, are supplementary added to the identifications of the original woodcut. The point **Q** bisects the frontal edge of the dodecahedron, the points **T**, **P**, **G**, and **F** represent a mirror line of the regular pentagonal face **ABCEF** which is shown in figure 3b in a larger size.

The points **T**, **F**, and **Q** define an intersecting plane which divides the dodecahedron in two mirror-inverted parts and gives the image area in figure 3c. Due to the fact that the dodecahedron center, **M**, the face center, **P**, and the dodecahedron corner, **F**, are elements of this image area, the representation of the circumscribed and inscribed radii is undistorted. The points **E**, **D**, **J**, and **A**, which define the inscribed cube in figure 3a, are written in parentheses because they are arranged in front of the image area, respectively behind it.

Although Kepler didn't present a calculation of the dodecahedron spheres, $\mathbf{S}_{in}^{dod}$ and $\mathbf{S}_{cir}^{dod}$, he specified the ratio of their radii, $\mathbf{r}_{cir}^{dod}/\mathbf{r}_{in}^{dod}$, as **100000** to **79465**. This result is obviously derived from a precise calculation.

In general, Kepler's calculations were written in prose, as usual in the late Renaissance. He mainly used the theorem of Pythagoras (570 – 495 BC), the rule of similar triangles and the rule of proportion.

For the reconstruction of his calculations here the two-point notation **AB** is used for both absolute value and straight line segment, in the same way as Kepler usually did. I.e., **AB** has the same (positive) value as **BA**.

For practical reasons the slash is used evenhandedly for ratios and divisions, and the square root, $\sqrt{\mathbf{x}}$, is denoted in the exponential form, $\mathbf{x}^{0.5}$, and the reciprocal value, **1/x**, in the exponential form, $\mathbf{x}^{-1}$**.**

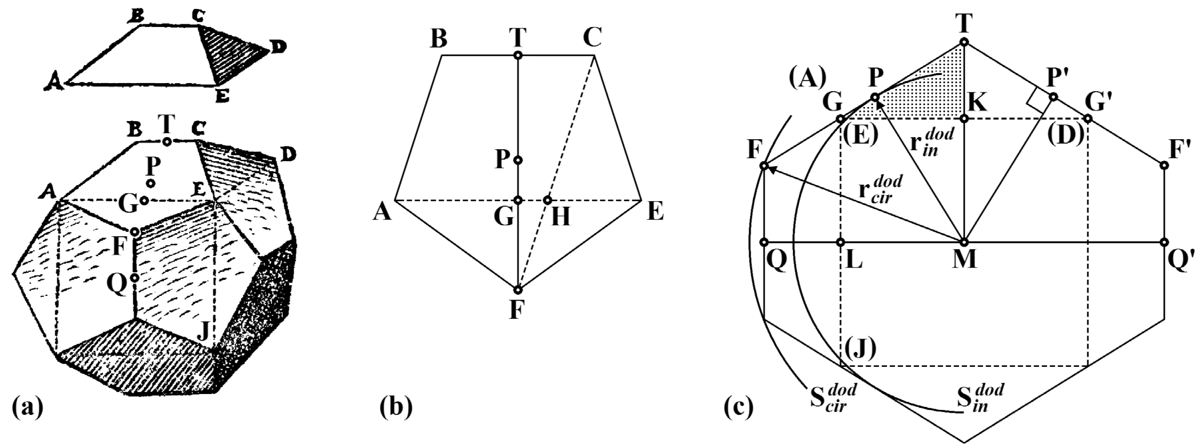

Figure 3. (a) Woodcut of a dodecahedron from *Harmonicis Lib. V, Cap. I.* (b) Regular pentagonal face **ABCEF** of Kepler's dodecahedron. The diagonal **AE** is an edge of the dotted outlined cube **AEDJ…** in the original woodcut. (c) Intersecting plane of the dodecahedron shown in figure 3a, seen from the right hand side, and defined by the points, **T**, **F**, and **Q**.

In figure 3b, the point **H** denotes the intersection point of the diagonals **AE** and **CF** of the regular pentagon **ABCEF**, dividing the diagonal **CF** into **CH** and **HF**. The ratio **CH/HF** is the *golden ratio* (lat. *proportio divina*), defined by the equation **CH/HF = CF/CH**. Today we commonly denote the resulting value by the greek letter, $\boldsymbol{\tau}$, with $\boldsymbol{\tau} = \mathbf{0.5}\,(5^{\,0.5} + 1) = \mathbf{1.61803398\ldots}$. Kepler knew how to calculate $\boldsymbol{\tau}$ with equal accuracy!

In figure 3b the rule of similar triangles leads from $\mathbf{CH/HF} = \boldsymbol{\tau}$ to $\mathbf{TG/GF} = \boldsymbol{\tau}$, and furthermore, in figure 3c, from $\mathbf{TG/GF} = \boldsymbol{\tau}$ to $\mathbf{ML/LQ} = \boldsymbol{\tau}$ and lastly to $\mathbf{MK/KT} = \boldsymbol{\tau}$, due to **MQ = MT** and **ML = MK**. In accordance to the previous definition of the *golden ratio* follows $\mathbf{MK/KT} = \mathbf{MT/MK} = \boldsymbol{\tau}$.

The rectangular triangles **GKT** and **MPT** are similar, due to their identical angles in **T**, although they are mirror-inverted. The rule of similar triangles leads to **MP/MT = GK/GT**, respectively to **MP = (GK/GT) MT**.

With $\mathbf{GT} = (\mathbf{GK}^2 + \mathbf{KT}^2)^{0.5} = (1^2 + \boldsymbol{\tau}^{-2})^{0.5}\,\mathbf{GK}$, and with $\mathbf{MT} = \boldsymbol{\tau}\,\mathbf{MK} = \boldsymbol{\tau}\,\mathbf{GK}$, results:

$\mathbf{r}_{in}^{dod} = \mathbf{MP} = (\mathbf{GK/GT})\,\mathbf{MT} = (1 / (1^2 + \boldsymbol{\tau}^{-2})^{0.5})\,\boldsymbol{\tau}\,\mathbf{GK} = \mathbf{1.376381920\ GK}$, and furthermore:

$\mathbf{r}_{cir}^{dod} = \mathbf{MF} = (\mathbf{MQ}^2 + \mathbf{QF}^2)^{0.5} = (\boldsymbol{\tau}^2 + \boldsymbol{\tau}^{-2})^{0.5}\,\mathbf{GK} = 3^{0.5}\,\mathbf{GK} = \mathbf{1.732050808\ GK}$ (note: $\mathbf{MF} = \mathbf{MA}_{undistorted}$).

The ratio of the radii of the inscribed to the circumscribed sphere is: $\mathbf{r}_{in}^{dod}/\mathbf{r}_{cir}^{dod} = \mathbf{0.\underline{79465}4472\ldots}$.

This reconstructed calculation confirms exactly the result of Kepler, $\mathbf{r}_{cir}^{dod}/\mathbf{r}_{in}^{dod}$, as **100000** to **<u>79465</u>**.

4. The reconstructed calculation of the spheres of the small stellated dodecahedron

The first known realisation of the small stellated dodecahedron, in the following simply called *the star*, is a marmor mosaic in the Basilica di San Marco in Venice from Paolo Uccello (1397 – 1475). Today the star is called a Kepler-Poinsot polyhedron. The star doesn't bear Kepler's name because of his famous woodcut shown in figure 4a, but rather in that he specified some exact measures of the star. He considered the inscribed sphere as the greatest possible one, and not, as we do today, as that one which is limited by the interpenetrating planes. This view is proved by the fact that Kepler marked the relating osculation points of the greatest inscribed sphere with the character **O**. The points **O** are the half-dividing points of the concave edges of the star respectively of the edges of the dodecahedron which is hidden inside the star (see figure 4a).

P and **P'** are corners of the star, i.e. the osculation points with its circumscribed sphere. They correlate to the corners of an icosahedron, which is the dual solid to a dodecahedron as shown in figure 4c. Kepler wrote in the original latin text [5]: '*…in stella, ut* **100000** *ad* **52573***, dimidium latus Icosaëdri, seu dimidium distantiæ duorum radiorum'*. I.e., Kepler calculated the ratio of the radii of the circumscribing sphere of the star to that inscribed in it as **100000** to **52573**. Furthermore he argues that the radius of the inscribed sphere is half the distance between two corners **P** and **P'** of the star respectively half the edge length of a relating icosahedron.

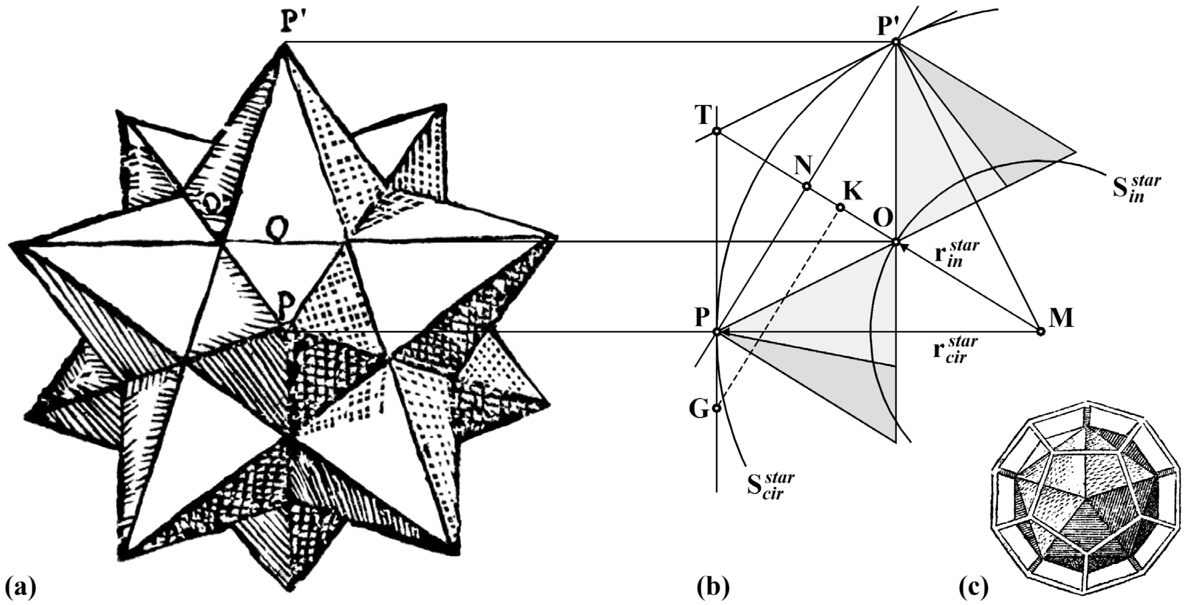

Figure 4. (a) Woodcut of the small stellated dodecahedron from *Harmonicis Lib. V, Cap. I.* (b) Intersecting plane defined by the points, **P**, **O**, and **P'**, seen from the right hand side of the star in figure 4a. The geometry of the points **P**, **O**, **P'**, **M**, **K**, **T**, and **G**, is equivalent to figure 3c, counterclockwise rotated about **M** with the angle **QMP**. (c) This original woodcut shows the duality of a dodecahedron to an icosahedron. The view is the same than in figure 4a, with **P** in the image center.

For a proof that the radius $\mathbf{r}^{star}_{in}$ is equal to **PP'/2** we have to show that **MO** is equal to **PN** (see figure 4b). With respect to the dotted outlined square in figure 3c it can be shown that the legs of the grey marked, right-angled triangle **GKT** have the ratio **GK/KT = τ**, by reason of **GK = MK** and **MK/KT = τ**. The similarity of the triangles **GKT** and **PNT** in figure 4b leads to the ratio **PN/NT = τ** and equally to **PN/NO = τ**, because **TP** and **P'O** are parallel. **PN/NO = τ** implies (**PN+NO**)/**PN** = **τ**, in accordance to the definition of the *golden ratio*.

The similarity of the triangle **PNT** to the triangle **MNP** involves **MN/PN = τ**. The substitution of **MN** with **MO+ON** leads to (**MO+ON**)/**PN** = **τ**.

The equalisation of (**MO+ON**)/**PN** = **τ** and (**PN+NO**)/**PN** = **τ** lastly gives **MO = PN**, q.e.d.

The radii are: $\mathbf{r}^{star}_{in} = \mathbf{MO}$ and $\mathbf{r}^{star}_{cir} = \mathbf{MP} = (\mathbf{MN}^2 + \mathbf{PN}^2)^{0.5} = ((\tau\,\mathbf{MO})^2 + \mathbf{MO}^2)^{0.5} = (\tau^2 + 1)^{0.5}\,\mathbf{MO}$

Consequently the ratio of the star radii amounts to: $\mathbf{r}^{star}_{in}/\mathbf{r}^{star}_{cir} = 1 / (\tau^2 + 1)^{0.5} = 0.\underline{52573}1112\ldots$.

In turn the calculation confirms the result of Kepler, with $\mathbf{r}^{star}_{cir}/\mathbf{r}^{star}_{in}$, as **100000** to **52573**.

5. Kepler’s dialectic understanding of artistic imagination and mathematical verification

In the *Mysterium Cosmographicum* [4] from 1596, Kepler illustrated the concentric spheres of the planets with a specific thickness (compare figure 1), to give space for the excentricity of the respective planetary orbits. The idea that the distances between the spheres of the planets are determined by the circumscribed and inscribed spheres of the five Platonic solids was probably inspired by the work of artists. The detail from Lorenz Stoer’s woodcut [6] in figure 5 shows an octahedron inside a perforated, enveloping sphere.

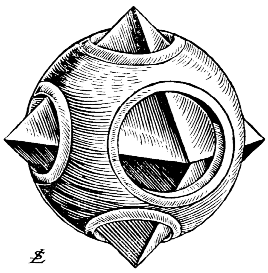

Figure 5. Lorenz Stoer, 1567
Detail from woodcut No. 8

Since 1600 Kepler lived in Prague, and worked as an assistant of the well-known Danish astronomer Tycho Brahe. In the years after Brahe’s death in 1602, Kepler discovered that the excentric orbits of the planets have elliptic shapes.

In 1609, in his work *Astronomia nova* [7], he published his first two laws about planetary motion on the base of Brahe’s famous astronomical data.

Both, Brahe’s data and Kepler’s increasing numeracy skills, let him revise his early model from 1596, but without completely discarding it. In the fifth book of his main work *Harmonices Mundi* [5], published in 1619, he remarked that there must act polymorphic geometrical figures beyond the five Platonic solids. He found out that the ratio of the distance Venus-Sun to the shortest distance Mars-Sun correlates to the ratio $\mathbf{r}_{in}^{star}/\mathbf{r}_{cir}^{star}$ (figure 4b), so that the intermediate sphere of the Earth would be interpenetrated by the twelve spikes of the star. Possibly, this idea was directly inspired by the illustration of Lorenz Stoer!

6. The relevance of Kepler’s work for the contemporary empirical science

Although Johannes Kepler was a religious man, he was one of the first who delivered the natural philosophy from the stranglehold of the ecclesiastic dogma, placing direct observation right at the top, in accordance to our present-day understanding of empirical science.

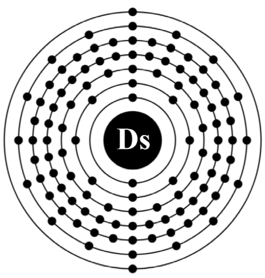

Figure 6. Nuclear shell model
Darmstadtium Ds 110 (1994)

On the one hand we admire the mathematical trueness of Kepler’s three laws about planetary motion, on the other hand we are temted to belittle his early cosmological model. However, this artistical model was, despite its falsity, the impetus of Kepler’s epoch-making astronomical work.

Today, we are challenged to imagine objects of the subatomic space. In that we cannot observe them directly we have to use simplifying models too. Figure 6 shows, in which way the 110 electrons of a synthesized Darmstadtium atom are distributed about seven shells in accordance to the nuclear shell model.

But just because it seems to be impossible to find a real representation of atomic substantiality, we should be aware that artistic imagination could be a helpful catalyst in an effort to transform measured data by complex calculations into a deeper insight.

Acknowledgements

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References

[1] David Wade, *Fantastic Geometry: Polyhedra and the Artistic Imagination in the Renaissance*, The Squeeze Press (2012)
[2] Luca Pacioli, *De divina Proportione*, Venice (1509)
[3] Wenzel Jamnitzer, *Perspectiva Corporum Regularium*, Nuremberg (1568)
[4] Johannes Kepler, *Mysterium Cosmographicum*, Tübingen (1596)
[5] Johannes Kepler, *Harmonices Mundi Libri V*, Frankfurt (1619)
[6] Lorenz Stoer, *Geometria et Perspectiva*, Augsburg (1567)
[7] Johannes Kepler, *Astronomia nova*, Prague (1609)