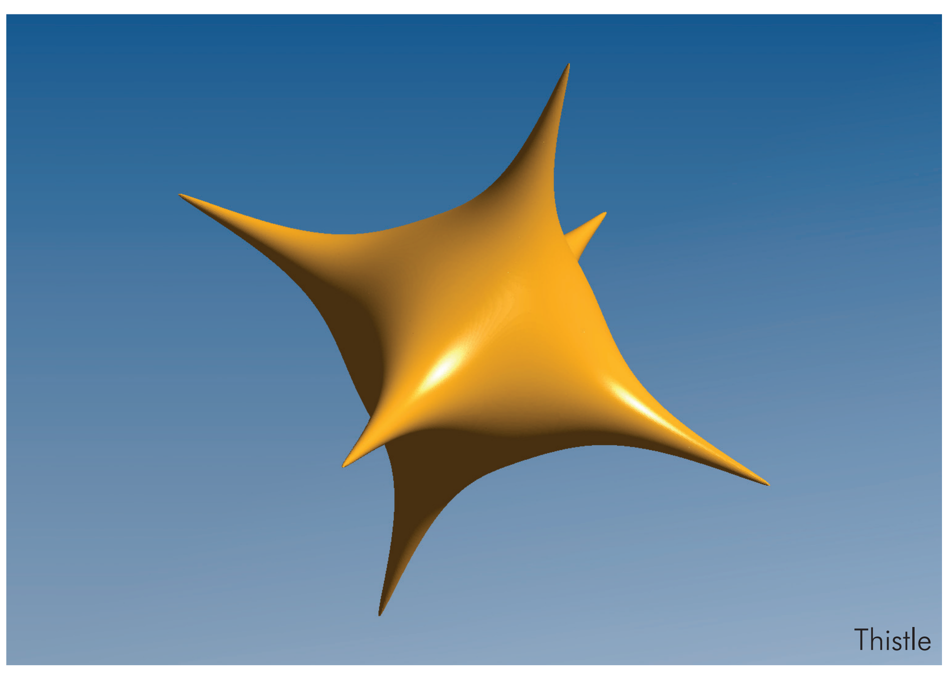
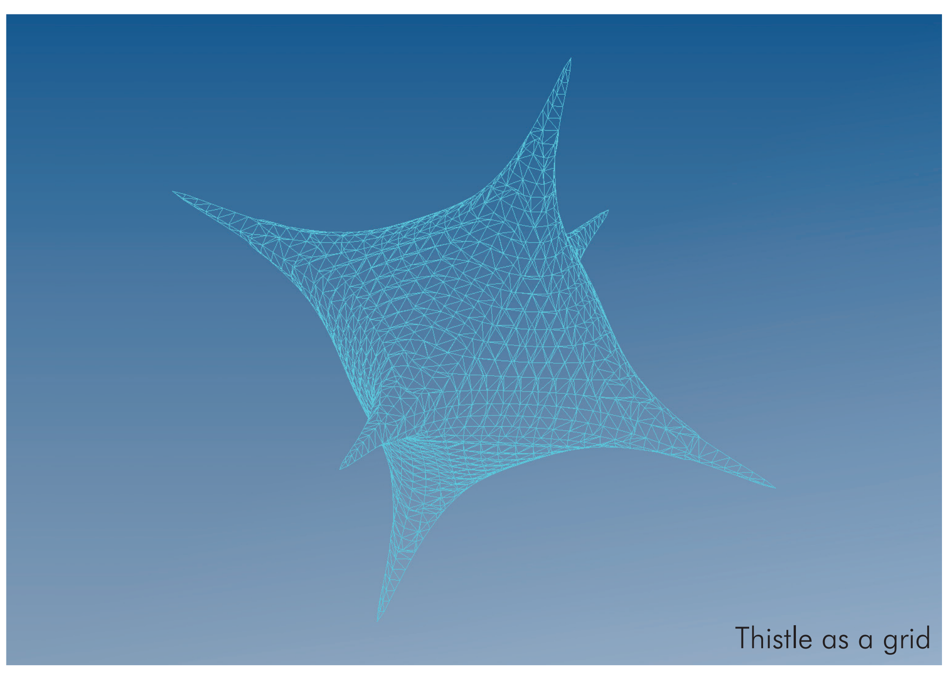
Sculptures

3D print of algebraic surfaces

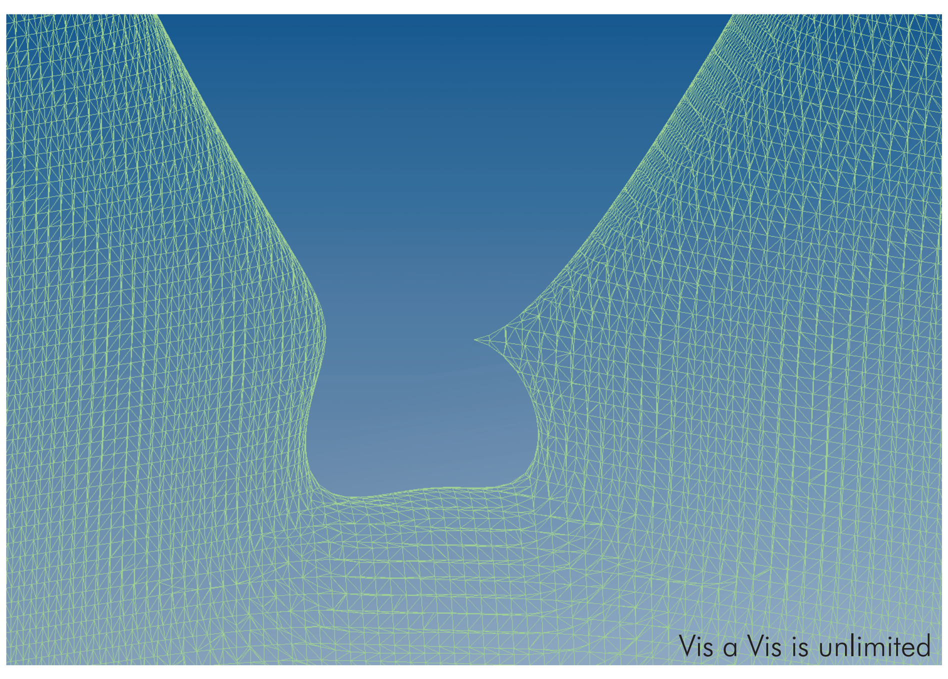
from the formula to the sculpture



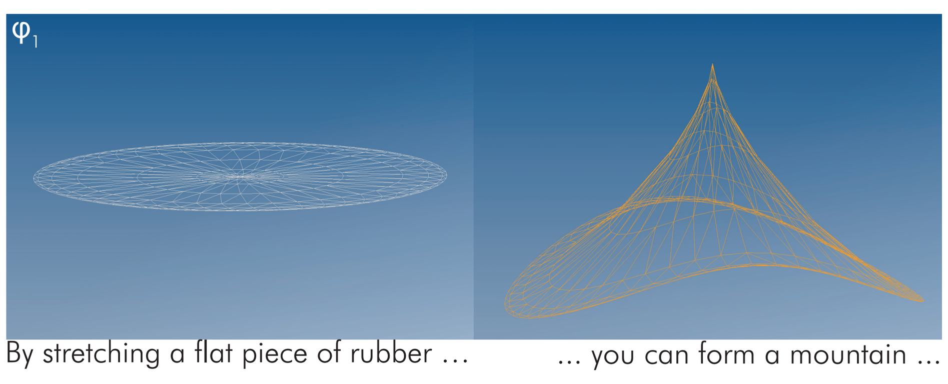
Thistle



Thistle as a grid

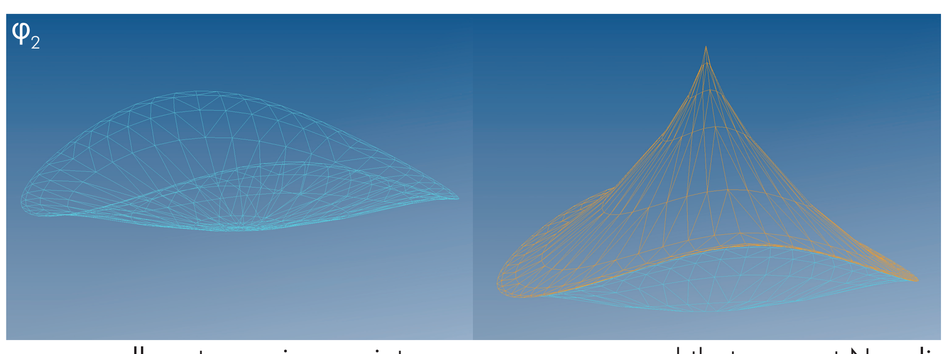


Vis a Vis is unlimited



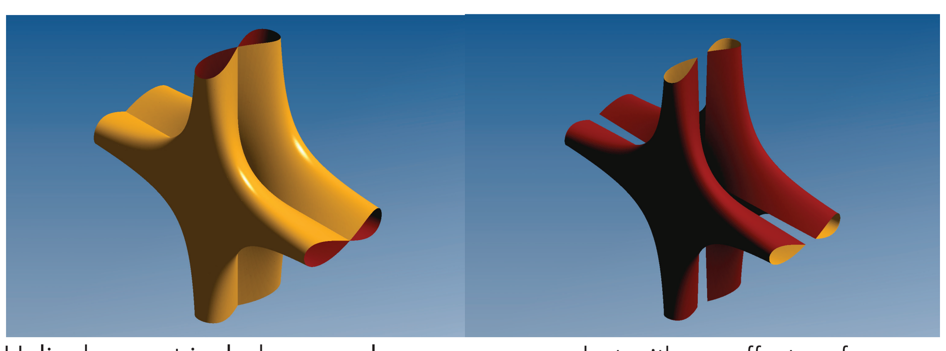
φ_1

By stretching a flat piece of rubber … … you can form a mountain …

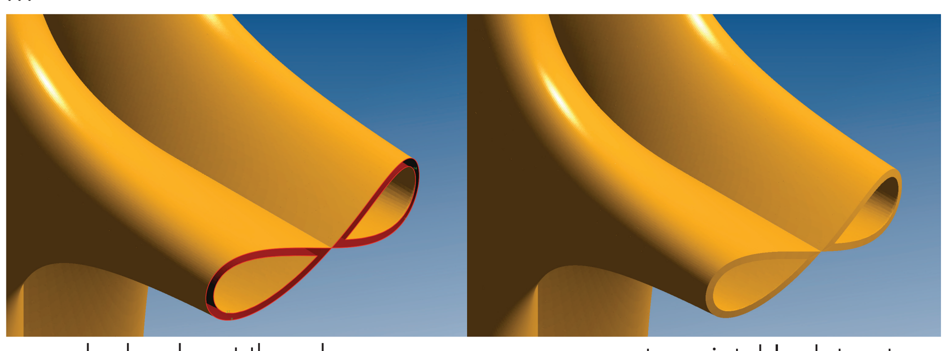


φ_2

… or a valley at any given point … … and that way get Nepali.

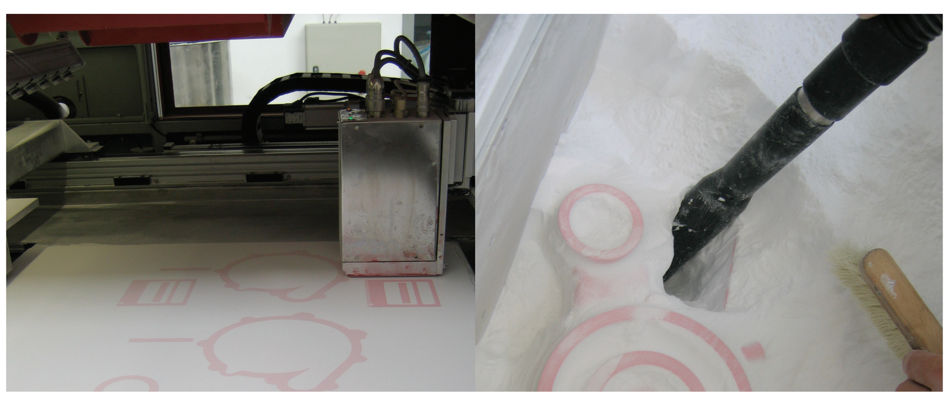


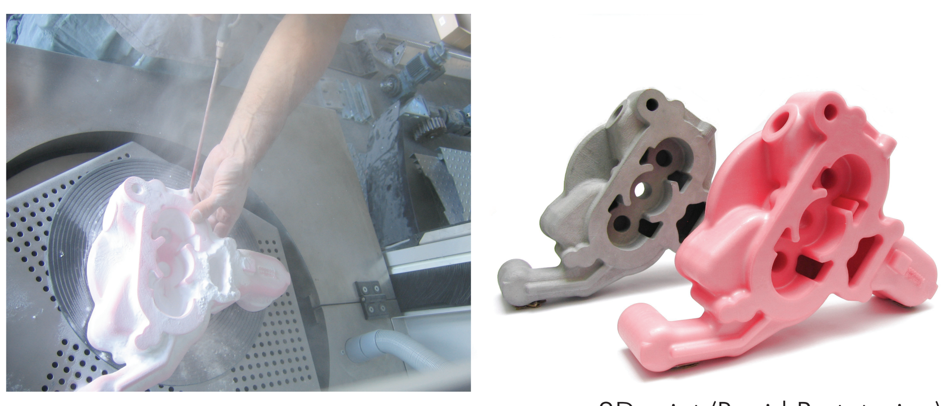
Helix does not include any volume, … … but with an offset surface, …



… and a border at the edges … … you get a printable dataset, …

… and finally a real sculpture.





3D print (Rapid Prototyping)

The various languages of algebraic surfaces and 3D-printers

Although algebraic surfaces consist of innumerable single points they can be concisely and elegantly described in full as zeros of a polynomial with few coefficients. However, a 3D printer needs other input data to establish a three-dimensional model. Here, virtual models are mostly transmitted as a network of triangles. All triangles of such a network, put together, include the volume of the component. To create curved bows they are approximated by thousands of little triangles. Therefore, if you wish to get a three-dimensional printout of algebraic surfaces, instead of a handful of coefficients of their polynomial, a network of triangles with several hundreds of thousands of coordinate points is needed. At FORWISS, the Institute for Software Systems in Technical Applications of the Information Technology Department of the University of Passau printable 3D data sets were created for some of the algebraic surfaces displayed at the IMAGINARY Exhibition.

Thinner than a soap bubble

Sculptures must be finitely large and have a volume. Algebraic surfaces, however, are infinitely thin and often have extend out to infinity. Some surfaces, such as Nepali, enclose a volume and this allows the computer to draw the surface enclosing this volume. Other surfaces such as Vis-A-Vis are unlimited - there is no interior or exterior. These are not the surfaces of any physical object and to create them as a visible sculpture is quite tricky.

The (in)finite search of the surface

The so-called marching cubes procedure is a known method that enables the approximate determination of the zero set of a function with three variables through a network of triangles. The space is divided into a grid of millions of small equal boxes. The function is evaluated only at the corners of these boxes instead of at infinitely many points. If the function values of all corners of a box have the same sign then the algorithm assumes that this box contains no piece of the surface searched for. Other boxes may have changes of sign. Hence, there must be a zero of the function here. A piece of the surface has been found. So, a net of triangles is drawn through the box to separate the positive values from the negative values. Taken together these separation surfaces for all boxes approximately equal the zero set of the function searched for. This way, the network of triangles required for the 3D-print can be calculated for some algebraic surfaces such as Dullo, Citric or Spinning top. The marching cube procedure is not suitable for all surfaces. Double changes of sign located close together often remain undetected. Sharp edges then reveal broken notches and gaps may also occur making surfaces disintegrate.

How mountains grow from rubber

In the case of some algebraic surfaces a different mathematical method can be used. The polynomial equation $f(x, y, z)=0$ can sometimes, e.g. in the case of Nepali, be solved for the variable z. Instead of the implicit description of the surface you then have (possibly several) explicit functions $z = \varphi_i(x,y)$. In the case of Nepali there are two functions, φ_1 and φ_2. Now, imagine a circular piece of rubber from a balloon lying flat on the ground. At each single point (x, y) of the rubber you can lift it up to the height $z=\varphi_1(x,y)$. That way a mountain arises in Nepali.
A second piece of rubber is pulled down to $z=\varphi_2(x,y)$ and a valley is obtained. The two pieces of rubber form the top and bottom side of Nepali. Rubber is elastic and if you skilfully select the points you can position the rubber for finitely many points. The rest of the rubber sheet will move by itself. A network of triangles reacts quite similarly.

A model from a single grain of sand and how to see a surface with no volume

The algebraic surfaces Helix and Vis-A-Vis have one thing in common: their zeros spread infinitely far away. There is no exterior nor interior in this case and no volume is enclosed. The trick to"see" the shape as the surface of a volume does not work here. Hence, to create a sculpture we cannot define the surface in terms of a net of triangles as before, because this would involve the 3D printer trying to create the whole model with less source material than a single grain of sand. For that reason the datasets of the exhibits shown here were calculated in such a way that the original surface was thickened towards one side. The marching cubes procedure is again used, but this time, the spatial distance of each box corner to the algebraic surface is calculated.

This distance measurement is important when checking whether a measured object deviates from its digital construction plan and shows deformations or distortions. Using digital scissors, the original surface and the "thickened"offset surface can be trimmed to fit. The gaps between the boundaries of the surfaces can be filled by stitching on triangles. While Vis-A-Vis has just one single circumferential boundary, Helix has eight pairs of boundary contours which have to be interconnected. After this procedure, the offset surface, boundary and algebraic surface describe a "waterproof" network of triangles and include a thin volume. The sculpture can then be printed.

Partner/sponsors: the sculptures were created for the exhibition by Alphaform and Voxeljet Technology. The 3D-data was compiled by FORWISS Institute of the University of Passau.

Mathematisches Forschungsinstitut Oberwolfach