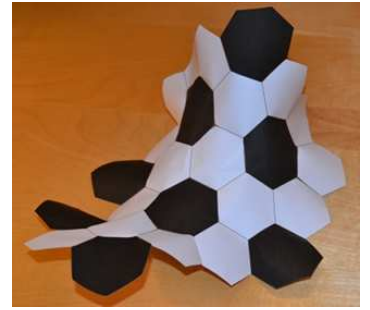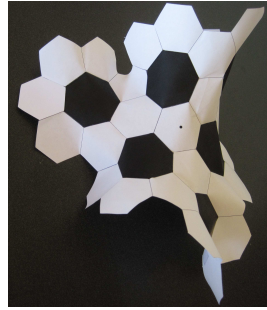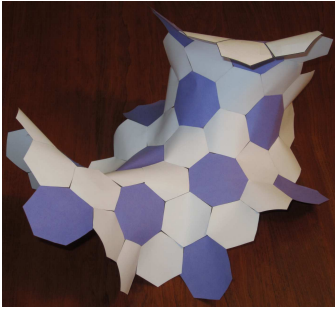# Hyperbolic Soccer Ball
## Frank Sottile

Constructing and studying the hyperbolic soccer ball is an enjoyable activity resulting in a beautiful model which can be used to investigate the hyperbolic plane and non-Euclidean geometry.
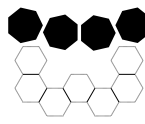


## Constructing the Hyperbolic Soccer Ball. These instructions construct the rightmost model above.

**Materials:**
- Printed templates. One:  One or two: 
- Scissors and Tape.
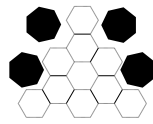
**Directions:** Carefully cut out *all* polygons along the dotted lines. Save the central hexagon in the first template, cutting a path to it. Tape a black heptagon edge-to-edge to the six hexagons in the hole, taping on the unprinted side. Place pieces of the tape across (*not* along) the edges, using two pieces for each edge, and taking care to *not* put tape over a vertex. When taping a second edge at a vertex, the model must be opened a bit (below left), and it will not lie flat. Tape the central hexagon into the gap which is left (below center). Tape two black heptagons
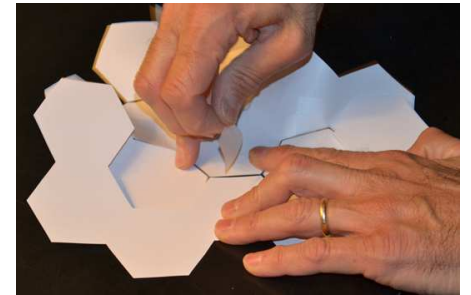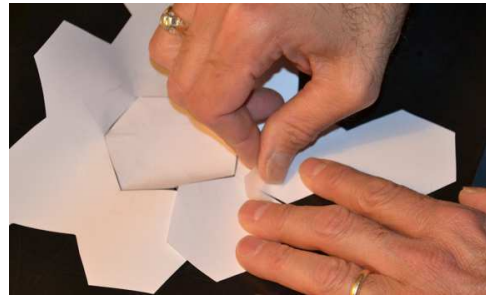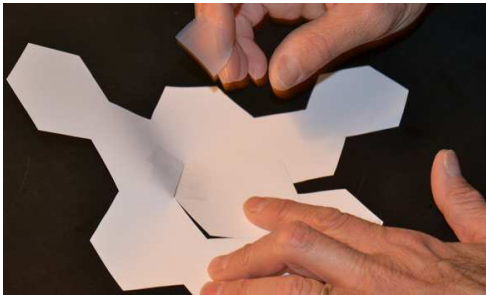


into the indentations along a side and then the W-shaped strip of hexagons from the second template to them (above right). This gives three heptagons surrounding a central hexagon, surrounded by 15 hexagons with one more breaking the symmetry. Tape the remaining five heptagons to complete this model (top right).

## Investigate Non-Euclidean Geometry.

**Materials:** Pencil and a short ($\leq$ 15 cm) straightedge, and a 3" × 5" card (card is not essential).

**Draw a line:** With unprinted side up, flatten two adjacent polygons and draw a line segment. Extend this segment across model with straightedge (or card) to a line $\ell$. Grasp the model with both hands (index fingers on the lines), pull taught, and sight down the line to verify that it is straight.
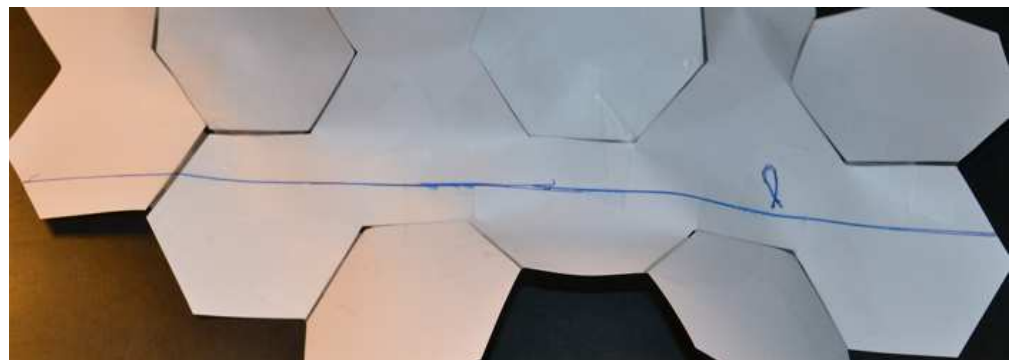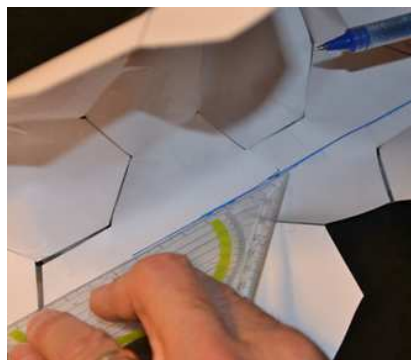
**Diverging parallels.** Draw a segment $m$ perpendicular to $\ell$, and then a line $\ell'$ perpendicular to $m$. The lines $\ell$ and $\ell'$ are parallel (lines perpendicular to the same line are parallel). Notice how they diverge.



**Parallel Postulate.** Given parallel lines, extend a new perpendicular segment from one ($\ell$) meeting the second at point $P$. The line perpendicular to the segment at $P$ is parallel to $\ell$, giving two parallels to $\ell$ through $P(\notin \ell)$. Thus Euclid's parallel postulate, or rather the equivalent Playfair's axiom (*At most one line can be drawn through any point not on a given line parallel to the given line in a plane*) does not hold. (This also constructs a Lambert quadrilateral (*three right angles*), but that is another story.) Compare your figure to that of your friends.



**Triangle.** Draw a large triangle on your model. What are its angles? You can measure their sum as follows: Cut a small semicircle from your card, marking a centre on the straight side. Put the centre at the vertex of an angle, align one ray along the edge, and mark the other ray ($A$) on the arc. Put the center at the vertex of a second angle, align one ray with the mark ($A$), mark the other ray ($B$) on the arc. Repeat for the third angle to get the angle-sum. Compare your angle-sum to that of others and try to formulate a conjecture for this angle-sum.
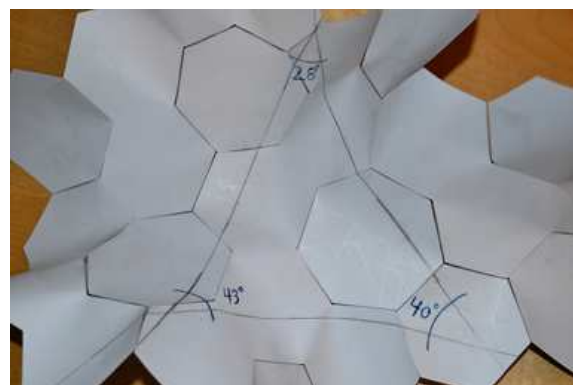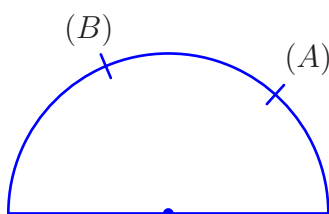




**Curvature.** The model is flat, except at its vertices, each of which contributes $-60/7° \approx -8.5°$ of curvature in a $\delta$-function. (This is the *excess* at the vertex, the difference of $360°$ and the sum of the incident angles.) This gives the deviation from Euclidean geometry. For your triangle check that

angle sum $+ 8.5° \cdot$ # vertices inside triangle $\approx 180°$.

For the triangle at right, we have $28 + 43 + 40 + 8 \cdot 8.5 = 179.0$. Check this relation, or better compare your angle-sum to that of other students. Do this also with your Lambert quadrilaterals. This makes a very fascinating guided discovery activity.
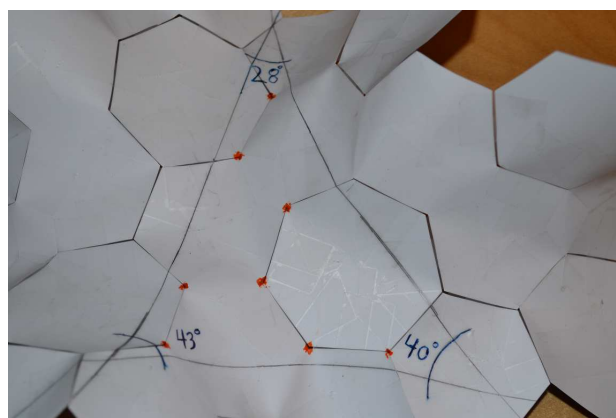


For more information, see `www.math.tamu.edu/~sottile/research/stories/hyperbolic_football/` or google "sottile hyperbolic football". I highly recommend making a very big model.