The Barth Sextic

This surface of degree 6 (sextic) was constructed by Wolf Barth in 1996. Altogether, it has 65 singularities if we include the 15 invisible ones which are infinitely far away. 65 is the maximum possible number of singularities on a sextic as shown in 1997 by Jaffe and Ruberman.

Barth's construction was a big surprise because until then geometers believed that a surface of degree 6 cannot have more than 64 singularities.

The icosahedral symmetry is one of the most striking features of Barth's sextic, but not all sextics with 65 singularities have this kind of symmetry; there is even a 3–parameter family of surfaces with 65 singularities!

In this family one can choose three parameters almost at random and always obtain a sextic with 65 singularities. The exact equation of Barth's sextic is $P_6 - \alpha K^2 = 0$, where $P_6 = (\tau^2x^2-y^2)(\tau^2y^2-z^2)(\tau^2z^2-x^2)$, $\tau = 1/2(1+\sqrt{5})$ is the golden ratio, $\alpha = 1/4(2\tau+1) = 1/4(2+\sqrt{5})$, and $K = x^2+y^2+z^2-1$ describes a sphere of radius 1.

Design: Oliver Labs

The Labs Septic

In 2004, while writing his Ph.D. thesis at Mainz, Oliver Labs constructed a surface of degree 7 (septic) with 99 singularities. Since 1997 it was known that for surfaces of degree 6 that the maximum possible number of singularities is 65.

In 1982, A.N.Varchenko discovered that in degree 7 there cannot be more than 104 singularities. In 1992, Chmutov constructed a septic with 93 singularities which was the world record at that time. Since Labs' construction, this world record has been 99. It is an open question whether septics can have 100, 101, . . . , 104 singularities!

The Labs Septic has the symmetry of a regular 7–gon. However, similar to Duco van Straten's computation for the sextics, one can compute that there is indeed a 5–parameter family of septics with 99 singularities. For the construction of the septic Oliver Labs used the computer algebra software Singular which was developed at the technical university of Kaiserslautern. This software is very well suited for working with singular algebraic surfaces.

Design: Oliver Labs

A quintic with 15 cusps

This surface of degree 5 (quintic) has 15 singularities whose type is called ordinary cusp or A2. The surface is part of a series of related surfaces of infinitely many degrees constructed by Oliver Labs in 2005.

As one can see from the picture, five of the singularities look different than the others. These five singularities are more specifically of type $A2^{++}$ and the others of type $A2^{+-}$. The former can be described locally by the equation $x^3+y^2+z^2 = 0$, the others by $x^3+y^2-z^2 = 0$.

The equation of the quintic with 15 cusps is $S_5(x, y)+t(z) = 0$, where $S_5(x, y) = x^5-10x^3y^2+5xy^4-5x^4-10x^2y^2-5y^4+20x^2+20y^2-16$ is a regular pentagon, and where the polynomial $t(z) = -3z^5+10z^3-15z-8$ is a variant of what are called Tchebychev Polynomials.

Design: Oliver Labs