DEFORMATIONS OF SYMMETRIC CMC SURFACES IN THE 3-SPHERE

SEBASTIAN HELLER AND NICHOLAS SCHMITT

Abstract. In this paper we numerically construct CMC deformations of the Lawson minimal surfaces $\xi_{g,1}$ using a spectral curve and a DPW approach to CMC surfaces in spaceforms.

1. Introduction

The moduli spaces of CMC (constant mean curvature) spheres and embedded CMC tori in the 3-sphere are well understood by now. The only CMC spheres are totally umbilic due to the vanishing of their Hopf differential. Brendle [3] recently proved the Lawson conjecture that the only embedded minimal torus in the 3-sphere is the Clifford torus. Using Brendles method Andrews and Li [1] have classified all embedded CMC tori in S^3. Additionally, all CMC immersions from a torus into 3-dimensional space forms are given rather explicitly in terms of algebro-geometric data on their associated spectral curves [22, 12, 2]. These integrable system methods are also applied to study the moduli space of all CMC tori, see for example [18, 20].

In contrast, higher genus CMC surfaces in S^3 are not very well understood. There are examples like the Lawson minimal surfaces [21] which exist for all genera. All known examples have been constructed by implicit methods from geometric analysis. However, there is no theory which describes the space of all CMC surfaces of higher genus, nor is there any classification of the embedded ones.

The study of CMC surfaces via integrable systems is based on the associated family

$$\lambda \in \mathbb{C}^* \mapsto \nabla^\lambda = \nabla + \lambda^{-1}\Phi - \lambda\Phi^*$$

of flat $\mathrm{SL}(2,\mathbb{C})$-connections on a fixed hermitian rank 2 bundle [12]. For minimal surfaces in S^3 the flatness of this family of connections is just a gauge theoretic reformulation of the Gauss-Codazzi and harmonic map equations. For CMC surfaces, the family of flat connections comes from the Lawson correspondence together with the Sym-Bobenko formula. The connections ∇^λ are unitary for $\lambda \in S^1 \subset \mathbb{C}^*$ and trivial at two Sym points $\lambda_1 \neq \lambda_2 \in S^1$. The immersion can be obtained as the gauge between ∇^{λ_1} and ∇^{λ_2}, and its mean curvature is given by $H = i\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}$. By loop group factorization methods, CMC surfaces can also be constructed out of families of flat connections which have a certain asymptotic behavior at $\lambda = 0$ and are unitarizable along the unit circle, i.e., unitary with respect to a λ-dependent metric (see Theorem 2).

Date: October 7, 2013.

These families of flat connections can be constructed by two different methods: the spectral curve approach and the DPW approach. The first describes the family via flat line bundles parametrized by a spectral curve, i.e., a double covering of the spectral plane, as in Theorem 4. The flat line bundles are defined on a double covering of our Riemann surface, and the moduli space of them is given by an affine bundle over the Prym variety. The second uses a so-called DPW potential [5], a loop of meromorphic $\mathfrak{sl}(2,\mathbb{C})$-valued 1-forms. The first method has the advantage that it is easier to deal with the unitarity condition, while the second can take advantage of the implementation of DPW in the XLab software suite.

The main difficulty in constructing higher genus CMC surfaces is that the generic connection ∇^λ is irreducible. Therefore, it is not understood by now how to make families of flat connections which are unitarizable along the unit circle. A flat connection is unitarizable if and only if its monodromy representation is unitary modulo conjugation. This is a condition which can be tackled numerically: using numerical ODE solvers one can compute the monodromy representation, and then apply basic results like Proposition 2 to determine whether a connection is unitarizable.

In the case of the spectral curve approach one also has theoretical support: as a consequence of the Narasimhan-Seshadri theorem it is known that for every holomorphic line bundle there exists exactly one flat compatible connection such that the corresponding flat $\mathrm{SL}(2,\mathbb{C})$-connection is unitary. This enables us to numerically determine the space of unitary connections. With this knowledge we can numerically search for families of flat connections which are unitarizable along the unit circle. With this spectral curve approach we reconstructed the Lawson surface $\xi_{2,1}$.

In the DPW approach, on the other hand, we combine these two steps, directly computing the families of unitarizable DPW potentials. The explicit translation from the spectral curve to the DPW theory provided initial data and elucidated the conditions at the sym points. We have carried out the DPW experiments for a special class of CMC surfaces, namely Lawson symmetric ones. They are equipped with a large group of extrinsic orientation preserving symmetries, which are holomorphic automorphisms on the Riemann surface. Due to this symmetry group, the moduli space of the possible Riemann surface structures is complex 1-dimensional. Its cotangent space is spanned by a quadratic differential which is the Hopf differential of a possible Lawson symmetric CMC immersion. A nice feature of such an immersion is that its curvature lines are closed (see Figure 1).

Our experiments give strong evidence to the existence of real 1-dimensional families of Lawson symmetric CMC surfaces passing through the Lawson surfaces $\xi_{g,1}$ themselves (see Figure 4). In the case of $g = 1$ this family is known from the spectral theory of CMC tori. We reconstructed this 1-parameter family numerically as a test of our procedure, bifurcating into the 2-lobed Delaunay tori of spectral genus 1, or continuing along the homogeneous tori of spectral genus 0. For higher genus Lawson symmetric CMC surfaces such bifurcations into higher spectral genus did not appear; these families continue until they collapse into double coverings of minimal spheres (as the Delaunay tori do). In genus 2 we have also found a family of Lawson symmetric CMC surfaces, disjoint from the family passing through $\xi_{2,1}$, which seems to converge

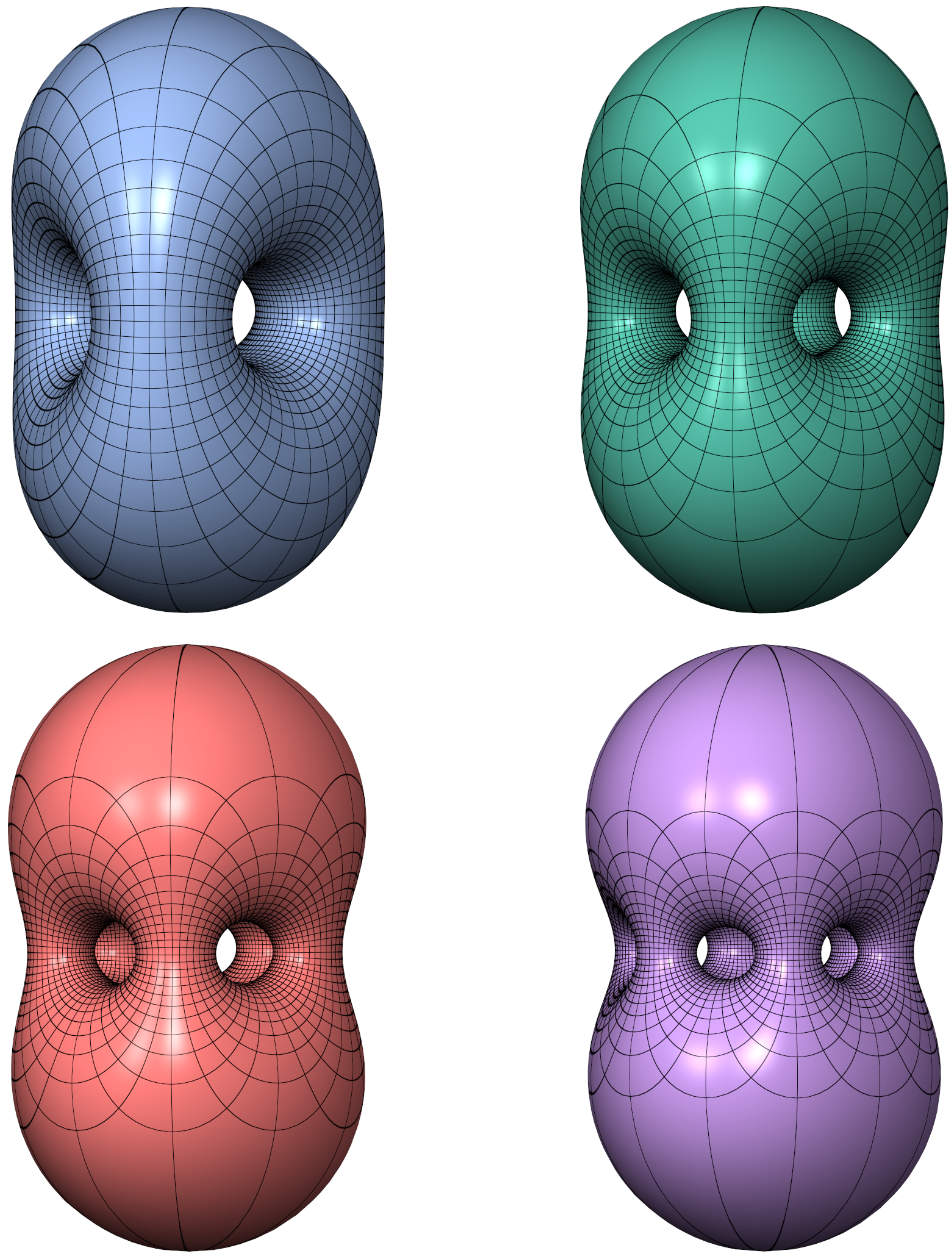

FIGURE 1. The Lawson surfaces $\xi_{g,1}$ of genus $g = 2, 3, 4, 5$.

to a threefold covering of a CMC sphere (see Figure 5). Altogether, our experiments begin to map out the moduli space of Lawson symmetric CMC surface of genus 2.

The paper is organized as follows: In chapter 2 we describe the necessary theory for our experiments. In chapter 3 we discuss the first experiments on the Lawson surface of genus 2 via the spectral curve approach. Chapter 4 concerns the numerical deformations of Lawson symmetric CMC surfaces of genus 2. Chapter 5 collects experiments with Lawson symmetric surfaces of higher genus. In the last chapter 6 we give a short outlook on the computational aspects of our studies.

2. Theoretical background

We shortly recall the well known description of conformal CMC immersions $f\colon M \to S^3$, where M is a Riemann surface and S^3 is equipped with its round metric [12, 2, 8]. Due to the Lawson correspondence, there is a unified treatment for all mean curvatures $H \in \mathbb{R}$:

Theorem 1. *Let $f\colon M \to S^3$ be a conformal CMC immersion. Then there exists an associated family of flat* $\mathrm{SL}(2,\mathbb{C})$*-connections*

$$\lambda \in \mathbb{C}^* \mapsto \nabla^\lambda = \nabla + \lambda^{-1}\Phi - \lambda\Phi^*$$

on a hermitian rank 2 bundle $V \to M$ which is unitary along $S^1 \subset \mathbb{C}^*$ and trivial at $\lambda_1 \neq \lambda_2 \in S^1$. Here, Φ is a nowhere vanishing complex linear 1-form which is nilpotent and Φ^* is its adjoint. Conversely, the immersion f is given as the gauge between ∇^{λ_1} and ∇^{λ_2} where we identify $\mathrm{SU}(2) = S^3$, and its mean curvature is $H = i\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}$. Therefore, every family of flat $\mathrm{SL}(2,\mathbb{C})$-connections satisfying the properties above determines a conformal CMC immersion.

Note that the complex linear part of the family of flat connections extends to $\lambda = \infty$ whereas the complex anti-linear part extends to $\lambda = 0$. It is well known [12], that for compact CMC surfaces which are not totally umblic, the generic connection ∇^λ of the associated family is not trivial. Moreover, for CMC immersions from a compact Riemann surface of genus $g \geq 2$, the generic connection ∇^λ of the associated family is irreducible [8].

An important observation is that it is often enough to work with a family connections which is only gauge equivalent (in a certain sense) to the associated family of a CMC surface. This enables us to use our preferred connections like meromorphic ones. In our situation we make use of the following theorem in order to construct compact CMC surfaces.

Theorem 2. *Let $U \subset \mathbb{C}$ be an open set containing the disc of radius $1 + \epsilon$. Let $\lambda \in U \setminus \{0\} \mapsto \tilde{\nabla}^\lambda$ be a holomorphic family of flat $\mathrm{SL}(2,\mathbb{C})$-connections on a rank 2 bundle $V \to M$ over a compact Riemann surface M of genus $g \geq 2$ such that*

- *the asymptotic at $\lambda = 0$ is given by*
$$\tilde{\nabla}^\lambda \sim \lambda^{-1}\Psi + \tilde{\nabla} + \ldots$$
where $\Psi \in \Gamma(M, K\,\mathrm{End}_0(V))$ is nowhere vanishing and nilpotent;
- *for all $\lambda \in S^1 \subset U \subset \mathbb{C}$ there is a hermitian metric on V such that $\tilde{\nabla}^\lambda$ is unitary with respect to this metric;*
- *$\tilde{\nabla}^\lambda$ is trivial for $\lambda_1 \neq \lambda_2 \in S^1$.*

Then there exists a unique (up to spherical isometries) CMC surface $f\colon M \to S^3$ of mean curvature $H = i\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}$ such that its associated family of flat connections ∇^λ and the family $\tilde{\nabla}^\lambda$ are gauge equivalent, i.e., there exists a λ-dependent holomorphic family of gauge transformations g which extends through $\lambda = 0$ such that $\nabla^\lambda \cdot g(\lambda) = \tilde{\nabla}^\lambda$ for all λ.

In the above form, this theorem was proven in [10], but there are earlier variants adapted to the DPW approach to k-noids [23, 6].

Remark 1. The theorem remains true, if there exists λ-independent apparent singularities of the connections $\tilde{\nabla}^\lambda$. It also remains true, if there exists finitely many point on the unit circle, where the monodromy is not unitary. In both cases, the corresponding singularities (on the Riemann surface in the first case, and on the spectral plane in the second) are captured in the positive part of the Iwasawa decomposition (see the proof of this theorem in [10]). Therefore, the actual associated family of flat connections ∇^λ has no singularities anymore, and the CMC immersion is well-defined.

From now on we focus on CMC immersions from a compact Riemann surfaces of genus 2 which have the following (extrinsic, space orientation preserving) symmetries:

- an involution φ_2 with exactly 6 fix points which is holomorphic on the surface and commutes with the other symmetries;
- a $\mathbb{Z}_3$-symmetry generated by φ_3 with 4 fix points which is also holomorphic on the surface;
- another holomorphic involution τ with only 2 fix points.

These surfaces are called Lawson symmetric CMC surfaces (of genus 2). The symmetries already fix the Riemann surface structure up to one complex parameter. To be more precise, the underlying Riemann surface is given by the equation

$$y^3 = \frac{z^2 - z_0^2}{z^2 - z_1^2}.$$

Clearly, Lawson symmetric Riemann surfaces corresponding to tuples $(z_0, z_1, -z_0, -z_1)$ with the same cross-ratio are isomorphic. The Riemann surface structure of the Lawson surface $\xi_{2,1}$ is given by $z_0 = 1$, $z_1 = i$. In this picture the symmetries are given on the Riemann surface by $\varphi_2(y, z) = (y, -z)$, $\varphi_3(y, z) = (e^{\frac{2}{3}\pi i} y, z)$ and $\tau(y, z) = ((\frac{z_0}{z_1})^{\frac{2}{3}} \frac{1}{y}, \frac{z_0 z_1}{z})$.

There is a method called dressing which makes new CMC surfaces out of old, see for example [4]. The idea is that a CMC surface is in general not uniquely determined by the family of gauge equivalence classes of its associated family of flat connections. It was shown in [10] that a dressing deformation of a Lawson symmetric CMC surface is not Lawson symmetric anymore. Therefore, for Lawson symmetric CMC surfaces it is enough to know the family of gauge equivalence classes of its associated family of flat connections.

Altogether, in order to find CMC surfaces we need to find a holomorphic curve in the moduli space of flat $\mathrm{SL}(2, \mathbb{C})$ connections on M which may be lifted to a family of flat connections satisfying the properties of Theorem 2. Moreover for Lawson symmetric CMC surfaces, we do not need to consider the moduli space of all flat $\mathrm{SL}(2, \mathbb{C})$ connections but only those which are equivariant with respect to φ_2, φ_3 and τ. We call these connections flat Lawson symmetric connections.

2.1. **The spectral curve approach.** One way to construct families of (gauge equivalence classes of) flat connections is based on Hitchin's abelianization [11]. We will restrict our discussion to the case of flat Lawson symmetric connections ∇. On a Riemann surface, a connection can be decomposed into a holomorphic and an anti-holomorphic structure

$$\nabla = \bar{\partial}^{\nabla} + \partial^{\nabla},$$

where $\bar{\partial}^{\nabla}$ maps to complex anti-linear 1-forms and ∂^{∇} maps to complex linear 1-forms. There are several reasons why it is useful to consider holomorphic structures in the discussion of flat connections on a (compact) Riemann surface: By the Narasimhan-Seshadri theorem there exists for a generic holomorphic structure on a degree 0 bundle a unique flat connection ∇ such that ∇ is unitary with respect to a suitable hermitian metric and such that $\bar{\partial}^{\nabla} = \bar{\partial}$. Second, if ∇ is already flat, and we add a (trace-free) complex linear 1-form $\Psi \in \Gamma(M, K\,\mathrm{End}_0(V))$ then $\nabla + \Psi$ is flat if and only if Ψ is holomorphic. Such 1-forms are called Higgs fields. This observation shows that the (moduli) space of flat connections is an affine bundle over the (moduli) space of holomorphic structures, where the fibers consist of the finite dimensional space of Higgs fields, at least at its smooth points. Moreover, in the generic fiber there is a unique point such that the corresponding flat connection is unitary for a suitable hermitian metric. And lastly, as we have already mentioned above, the family of holomorphic structures $\bar{\partial}^{\nabla^{\lambda}}$ extends to $\lambda = 0$. Therefore it seems to be very useful to discuss the moduli space of flat connections as an affine bundle over the moduli space of holomorphic structures in order to analyze the asymptotic behavior of ∇^{λ} for $\lambda \to 0$.

As we are only interested in Lawson symmetric connections, we only need to deal with Lawson symmetric Higgs fields, i.e., Higgs fields which are also equivariant with respect to φ_2, φ_3 and τ. It was shown in [10] that for a generic Lawson symmetric holomorphic structure $\bar{\partial}^{\lambda}$, the Lawson symmetric Higgs fields constitute a complex line. Their determinant is a holomorphic quadratic differential and invariant under the symmetries. Therefore, for a generic Lawson symmetric holomorphic structure and a non-zero Lawson symmetric Higgs field Ψ its determinant $\det \Psi$ is a non-zero multiple of the pull-back of $\frac{dz^2}{(z^2-z_0^2)(z^2-z_1^2)}$. Its zeros are simple, so the eigenlines of Ψ are only well defined on a double covering $\pi\colon \tilde{M} \to M$. Clearly, $\tilde{M}$ inherits the symmetries of M. Note that $\tilde{M}/\mathbb{Z}_3$ is a torus while $M/\mathbb{Z}_3$ is the projective line. The eigenlines $L_{\pm}$ of Lawson symmetric Higgs fields with non-zero determinant satisfy $L_+ \otimes L_- = \pi^* K_M$. Therefore, the eigenlines for all those Lawson symmetric Higgs fields constitute an affine Prym variety for $\tilde{M} \to M$. As a base point of this affine Prym variety we fix the pull-back of the dual of the (unique) Lawson symmetric spin bundle $S^* = L(-Q_1 - Q_3 + Q_5) \to M$, where the points Q_1, Q_3 and Q_5 are Weierstrass points which make an orbit under the $\mathbb{Z}_3$-action. This enables us to understand the moduli space of Lawson symmetric holomorphic structures $\bar{\partial}^{\lambda}$.

Proposition 1. [10] *There exists an even holomorphic map*

$$\Pi\colon Jac(\tilde{M}/\mathbb{Z}_3) \to \mathcal{S} = \mathbb{P}^1 \tag{2.1}$$

of degree 2 to the moduli space $\mathcal{S}$ of Lawson symmetric holomorphic bundles. This map is determined by $\Pi(L) = [\bar\partial]$ for $L \neq \underline{\mathbb{C}} \in Jac(\tilde M/\mathbb{Z}_3)$ such that $\pi^* S^* \otimes \tilde\pi^* L$ is isomorphic to an eigenline bundle of a symmetric Higgs field of the Lawson symmetric holomorphic rank two bundle $(V, \bar\partial)$. The branch points of Π are the spin bundles of $\tilde M/\mathbb{Z}_3$ and the branch images of the non-trivial spin bundles are exactly the isomorphism classes of the strictly semi-stable holomorphic bundles, i.e., the corresponding unitary flat connections are reducible.

Away from the zeros of $\det\Psi$, the eigenlines of a Lawson symmetric Higgs field Ψ with respect to a Lawson symmetric holomorphic structure $\bar\partial$ span the holomorphic rank 2 bundle π^*V, i.e., there is a holomorphic map $\phi\colon L_+ \oplus L_- \to \pi^*V$ which is an isomorphism away from the zeros. A flat connection ∇ with $\bar\partial^\nabla = \bar\partial$ can be pulled back to $L_+ \oplus L_- \to \tilde M$ in order to yield a meromorphic connection also denoted by ∇. The second fundamental forms of ∇ with respect to the eigenlines are meromorphic line bundle valued 1-forms, and the residuum of ∇ at the zeros of $\det\Psi$ can be easily computed. Adding a multiple of the Higgs field Ψ to ∇ on V corresponds to adding a diagonal 1-form to ∇ on $L_+ \oplus L_-$. In our Lawson symmetric situation, the connection ∇ on $L_+ \oplus L_-$ is given explicitly in terms of theta-functions on the torus $\tilde M/\mathbb{Z}_3$. But it is even easier to work on the quotient of $\tilde M/\mathbb{Z}_3$ by the symmetries φ_2 and τ which a again is a torus, denoted by T^2. We will only state the formulas in the case of the Lawson Riemann surface structure. In this case $\tilde M/\mathbb{Z}_3$ as well as T^2 are square tori. If we identify $T^2 = \mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ then a Lawson flat symmetric connections corresponds to the connection 1-form

$$\omega = \omega_{x,a} = \begin{pmatrix} \pi a dz - \pi x d\bar z & c\frac{\theta(z-2x)}{\theta(z)} e^{-4\pi i x \operatorname{Im}(z)} dz \\ c\frac{\theta(z+2x)}{\theta(z)} e^{4\pi i x \operatorname{Im}(z)} dz & -\pi a dz + \pi x d\bar z \end{pmatrix} \tag{2.2}$$

where θ is the theta-function of T^2 which has a simple zero at 0 and

$$c = \frac{1}{6}\sqrt{\frac{\theta'(0)^2}{\theta(2x)\theta(-2x)}}. \tag{2.3}$$

The corresponding holomorphic structure $\bar\partial^\nabla$ on the rank 2 bundle is determined by $\Pi(\bar\partial^0 \pm \pi x d\bar z)$, and adding a multiple of the Higgs fields on ∇ is equivalent to adding a multiple of the diagonal matrix with entries dz and $-dz$ on ω. This discussion also leads to a full understanding of the moduli space of Lawson symmetric flat connections:

Theorem 3. [10] *Let $\bar\partial$ be a Lawson symmetric semi-stable holomorphic structure on a rank 2 vector bundle over M. Assume that $\bar\partial$ is determined by the non-trivial holomorphic line bundle $L \in Jac(\tilde M/\mathbb{Z}_3)$, i.e., $\Pi(L) = [\bar\partial]$. Then there is a 1:1 correspondence between holomorphic connections on $L \to \tilde M/\mathbb{Z}_3$ and flat Lawson symmetric connections ∇ with $\nabla'' = \bar\partial$. The correspondence is given explicitly by the connection 1-form (2.2).*

The remaining flat Lawson symmetric connections are given by two lines lying over the point $\underline{\mathbb{C}} \in Jac(\tilde M/\mathbb{Z}_3)$. For this case $x = 0$, and formula 2.2 breaks down. This is not surprising, since the holomorphic structure corresponding to $\underline{\mathbb{C}} \in Jac(\tilde{M}/\mathbb{Z}_3)$ is the holomorphic direct sum $S^* \oplus S \to M$ which does not provide a flat connection. Nevertheless, the gauge orbit of this holomorphic structure is infinitesimal close to the gauge orbits of two other holomorphic structures, namely the holomorphic structure corresponding to the uniformization of the Riemann surface (which does not provide a unitary flat connection) and the holomorphic structure $\bar{\partial}^{\nabla^0}$ given by the (well-defined) limit of $\bar{\partial}^{\nabla^\lambda}$ for $\lambda \to 0$ of the associated family. Both holomorphic structures admit an affine line of Lawson symmetric flat connections, and they are given as special limits of (2.2), see [10] for details.

Theorem 3 shows that flat Lawson symmetric connections are uniquely determined by a flat line bundle connection on $\tilde{M}/\mathbb{Z}_3$. But this flat line bundle is not unique as its dual gives rise to the same flat $\mathrm{SL}(2,\mathbb{C})$-connection. Therefore, in order to parametrize families of flat connections $\lambda \in \mathbb{C}^* \to \nabla^\lambda$, one needs in general a double covering $\Sigma \to \mathbb{C}^*$ in order to parametrize the corresponding family of flat line bundles. This leads to the following picture:

Theorem 4. [10] *Let $\lambda \mapsto \nabla^\lambda$ be the associated family of a conformal Lawson symmetric CMC immersion of a compact Riemann surface of genus 2. Then there exists a Riemann surface $p\colon \Sigma \to \mathbb{C}$ double covering the spectral plane $\mathbb{C}$ together with a map $\mathcal{L}\colon \Sigma \to Jac(\tilde{M}/\mathbb{Z}_3)$ such that*

$$\begin{tikzcd}
\Sigma \arrow[r, "\mathcal{L}"] \arrow[d, "p"] & Jac(\tilde{M}/\mathbb{Z}_3) \arrow[d, "\Pi"] \\
\mathbb{C} \arrow[r, "{[\bar{\partial}^\lambda]}"'] & \mathcal{S}
\end{tikzcd}$$

commutes. The spectral curve Σ branches at 0. Moreover, there exists a meromorphic lift $\mathcal{D}$ with a first order pole over $\lambda = 0$ into the affine moduli space $\mathcal{A}^f$ of flat line bundles on $\tilde{M}/\mathbb{Z}_3$ such that

$$\begin{tikzcd}
 & \mathcal{A}^f \arrow[d, "''"] \arrow[ddr, "abel"] & \\
\Sigma \arrow[ur, "\mathcal{D}"] \arrow[r, "\mathcal{L}"'] \arrow[d, "p"] & Jac(\tilde{M}/\mathbb{Z}_3) & \\
\mathbb{C} \arrow[rr, "{[\nabla^\lambda]}"] & & \mathcal{A}_2^f
\end{tikzcd}$$

commutes, where $\mathcal{A}_2^f$ is the moduli space of flat Lawson symmetric connections on M and abel is the map discussed in Theorem 3.

Conversely, a triple $(\Sigma, \mathcal{L}, \mathcal{D})$ as above determines a family of Lawson symmetric flat connection on M which has the asymptotic behavior as in Theorem 2. In order to obtain a CMC immersion the family of flat connections has to satisfy the reality condition and the closing condition. The second condition is easy compared to the first one as one knows which flat line bundle on the torus $\tilde{M}/\mathbb{Z}_3$ determines the trivial connection on M: It is the flat unitary line bundle which has monodromy

-1 along both of the "standard" generators of the first fundamental group of the torus $\tilde{M}/\mathbb{Z}_3$. The main difficulty is to find spectral data $(\Sigma, \mathcal{L}, \mathcal{D})$ which satisfy the reality condition, i.e., the corresponding family of flat $\mathrm{SL}(2,\mathbb{C})$-connections must be unitarizable along the unit circle. We work out the necessary theory to attack this problem numerically. As we have discussed above, for each (Lawson symmetric) holomorphic structure, there exists a unique compatible flat (Lawson symmetric) $\mathrm{SL}(2,\mathbb{C})$-connection which is unitarizable, i.e., unitary with respect to a suitable chosen metric. Clearly, this property is equivalent to have unitarizable monodromy. From Theorem 3 we see that for each holomorphic line bundle on the torus $\tilde{M}/\mathbb{Z}_3$ there is a compatible flat connection such that the corresponding flat Lawson symmetric $\mathrm{SL}(2,\mathbb{C})$-connection is unitarizable. Therefore, we obtain a (real analytic) section

$$a^u \in \Gamma(Jac(\tilde{M}/\mathbb{Z}_3), \mathcal{A}^F)$$

of the affine moduli space of flat line bundles over the Jacobian. With the same notations as used in (2.2) this section is given in the case of the Lawson Riemann surface by

$$a^u(x) = -\frac{1}{12\pi}\frac{\theta'(-2x)}{\theta(-2x)} + \frac{1}{12\pi}\frac{\theta'(2x)}{\theta(2x)} + \frac{1}{3}x + \frac{2}{3}\bar{x} + b(x), \tag{2.4}$$

where $b\colon Jac(\tilde{M}/\mathbb{Z}_3) \to \mathbb{C}$ is a doubly periodic real-analytic function. This function can easily be approximated to arbitrary order, see section 3.

The reality condition can now be rephrased as follows: For all $\mu \in \Sigma$ with $p(\mu) \in S^1$ the spectral data have to satisfy

$$a^u(\mathcal{L}(\mu)) = \mathcal{D}(\mu). \tag{2.5}$$

We will use this equation later on to determine the spectral data of the Lawson surface of genus 2 numerically, see Figure 3.

2.2. **The DPW approach.** Another approach to CMC surfaces in S^3 was developed by Dorfmeister, Pedit and Wu [5]. The basic idea is to work with families of meromorphic connections with respect to the trivial holomorphic rank 2 bundle $\underline{\mathbb{C}}^2 \to M$ instead of using the varying holomorphic structures $\bar{\partial}^\lambda$. Clearly, one needs to allow poles in the connection 1-forms as the only holomorphic unitarizable connection on $\underline{\mathbb{C}}^2 \to M$ over a compact Riemann surface is the trivial one.

In order to construct CMC surfaces one tries to find a DPW potential

$$\eta = \eta(\lambda) = \lambda^{-1}\eta_{-1} + \eta_0 + \eta_1\lambda + ..,$$

i.e., a meromorphic λ-family of meromorphic $\mathfrak{sl}(2,\mathbb{C})$-valued 1-forms on M with first order pole in λ such that the corresponding family of flat connections $\nabla^\lambda = d + \eta(\lambda)$ satisfies the properties of Theorem 2. In general, the DPW potential η does not exist on the whole spectral plane $\mathbb{C}^*$ but only on a small punctured disk around $\lambda = 0$. Moreover, it is not clear in general how many (possibly varying) poles one needs to allow in order to obtain a potential which give rise to a closed CMC surface in S^3. In the case of the Lawson surface of genus 2, the existence and precise form up to two unknown functions in λ of such a potential was determined in [9]. In the more general situation of Lawson symmetric CMC surfaces on a Riemann surface given by the equation

$$y^3 = \frac{z^2 - z_0^2}{z^2 - z_1^2} \tag{2.6}$$

one can easily prove by the same methods that a DPW potential is given by

$$\eta = \eta_{A,B} = \pi^* \begin{pmatrix} -\frac{2}{3}\frac{z(2z^2-z_0^2-z_1^2)}{(z^2-z_0^2)(z^2-z_1^2)} + \frac{A}{z} & \lambda^{-1} - \frac{(A+\frac{2}{3})(A-\frac{1}{3})}{B} z^2 \\ \frac{B}{(z^2-z_0^2)(z^2-z_1^2)} - \frac{\lambda A(A+1) z_0^2 z_1^2}{z^2(z^2-z_0^2)(z^2-z_1^2)} & \frac{2}{3}\frac{z(2z^2-z_0^2-z_1^2)}{(z^2-z_0^2)(z^2-z_1^2)} - \frac{A}{z} \end{pmatrix} dz. \tag{2.7}$$

Here, A, B are λ-dependent holomorphic functions on a neighborhood of $\lambda = 0$ and $\pi\colon M \to M/\mathbb{Z}_3 = \mathbb{CP}^1$. All poles are apparent on M, i.e. the local monodromy around every pole is trivial. On the quotient $M/\mathbb{Z}_3 = \mathbb{CP}^1$ the poles at $z = 0$ and $z = \infty$ are still apparent whereas the conjugacy class of the monodromy around the poles at the four branch points $\pm z_0$ and $\pm z_1$ is given by the third root of the identity.

The functions A and B need to be chosen in such a way that the closing condition and the reality condition is satisfied for the family of flat connections $d + \eta(\lambda)$. As was proven in [9] there do not exists finite values for A, B and λ such that the holonomy of $d + \eta_{A,B}$ is trivial. Nevertheless, there exists values for A and B such that the monodromy is upper triangular, and these values will guarantee our closing condition. The reason behind this is that the gauge from the associated family of flat connections to the connections $d + \xi_{A(\lambda),B(\lambda)}$ is singular at the Sym points. This can be deduced by comparing the spectral curve approach with the DPW approach: As we have two different ways to describe Lawson symmetric flat $\mathrm{SL}(2,\mathbb{C})$-connections there must exist a transformation between them. This transformation

$$(x, a) \mapsto (A(x, a), B(x, a)) \tag{2.8}$$

satisfies that the connections $d+\omega_{x,a}$ and $d+\eta_{A(x,a),B(x,a)}$ are gauge equivalent whence pulled back to $\tilde{M}$. It can be computed explicitly in terms of theta functions of the torus $\tilde{M}/\mathbb{Z}_3$. The gauge gets singular at the trivial connection but the transformation holomorphically extends through the corresponding values of x and a. As a consequence, the corresponding meromorphic connection $d + \eta_{A(x,a),B(x,a)}$ on the 4-punctured projective line has only upper triangular monodromy and not a diagonal one. Using this observation we obtain the following generalized extrinsic closing conditions at the Sym pointsfor Lawson symmetric CMC surfaces of genus 2 : The functions A and B are related at the Sym points λ_1 and λ_2 by

$$B(\lambda_k) = S_k(\lambda_k) \quad \text{and} \quad B'(\lambda_k) = S_k'(\lambda_k) \tag{2.9}$$

where

$$S_k(\lambda) = z_k^2 \lambda R(\lambda) \quad \text{with} \quad R(\lambda) = A(\lambda)(A(\lambda) - \frac{1}{3}). \tag{2.10}$$

One can easily verify by hand that for functions A and B satisfying the above equations the flat connections $d+\xi A(\lambda_k), B(\lambda_k)$ have upper triangular monodromy. Note that Theorem 2 can still be applied, see Remark 1 or [23, 6].

For our numerical computations, we do not work with the DPW potential $d + \eta_{A,B}$ as it has a singularity at $z = 0$. One can easily gauge this apparent singularity away by the gauge $\begin{pmatrix} 1 & 0 \\ -\frac{A\lambda}{z} & 1 \end{pmatrix}$ to obtain a meromorphic potential $d+\tilde{\eta}_{A,B}$ which is smooth at $z=0$. Moreover, it satisfies

$$\varphi_2^*(d+\tilde{\eta}_{A,B}) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}(d+\tilde{\eta}_{A,B})\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

where φ_2 on $\mathbb{CP}^1$ is given by $z \mapsto -z$. This implies that at $z=0$ the monodromy matrices M_1, M_2, M_3 and M_4 around the poles z_0, z_1 $-z_0$ and z_1 with respect to the standard basis of $\mathbb{C}^2$ are related as follows

$$M_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} M_1 \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \text{ and } M_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} M_2 \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

All these matrices are in $\mathrm{SL}(2,\mathbb{C})$ and of trace -1 as the singularities are apparent when pulled back to the threefold covering $M \to \mathbb{CP}^1$. We denote the traces of the products by

$$2t_{i,j} = \mathrm{tr}(M_iM_j).$$

The following Proposition gives an easy characterization of unitarizable representations which we apply in our experiments below:

Proposition 2. *Let the four matrices M_k be given as above such that they have no common eigenline. Then they are simultaneously unitarizable if and only if $t_{k,l} \in [-1,1]$ for all $k,l \in \{1,..,4\}$. This condition already holds if $t_{1,2} \in (-1,1)$ and $t_{1,3} \in (-1,1)$. In this case, the four matrices are unitarizable by a diagonal matrix.*

3. Experiments: The Lawson surface $\xi_{2,1}$

As we have described in section 2.1 we need to find a family of flat line bundles on the torus $\tilde{M}/\mathbb{Z}_3$ parametrized on the spectral curve $\Sigma \to \mathbb{C}$ which satisfies the reality condition (2.5) in order to construct the Lawson surface $\xi_{2,1}$. To do so, we first determine the set of unitary connections numerically, i.e., we compute the doubly-periodic function b in (2.4): For each $x \in \mathbb{C}$ and the holomorphic line bundle $\bar{\partial}^0 - \pi x d\bar{z}$ on $\tilde{M}/\mathbb{Z}_3$ we searched for the unique $a^u(x)$ such that the monodromy of the corresponding flat Lawson symmetric $\mathrm{SL}(2,\mathbb{C})$-connection is unitarizable. An irreducible flat connection is unitarizable if and only if the traces of all its individual monodromies are contained in the interval $[-2,2] \subset \mathbb{R}$. This leads naturally to a functional depending on a which can be numerically minimized by using a numerical ODE solver as implemented for example in *Mathematica*. This procedure was done for all points x in the torus $Jac(\tilde{M}/\mathbb{Z}_3)$ lying on a grid. The doubly-periodic function b, which is the difference between a^u and an explicitly known expression (2.4), can then be approximated by Fourier series on the Jacobian. For the Lawson Riemann surface the real part of the function b is shown in Figure 2 whereas its imaginary part is given via the formula $b(ix) = -ib(x)$ due to a real symmetry of the Lawson surface. Equipped with these numerical data, we searched for the spectral data of the Lawson surface. We have started with the assumption that the spectral curve does not branch over the closed punctured unit disc $\{\lambda \in \mathbb{C} \mid 0 < \lambda\bar{\lambda} \neq 1\}$. This assumption seems to be natural in view of the assertion concerning the branch points in Theorem 5 in

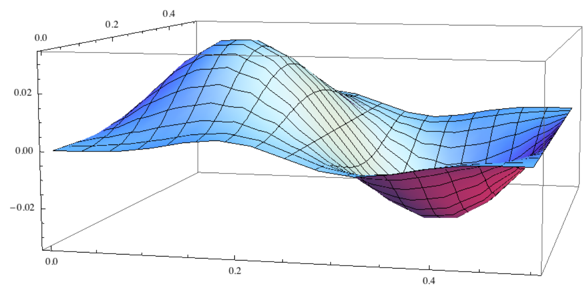

FIGURE 2. The real part of the function b

[10]. Then, an appropriate coordinate on Σ is given by t with $t^2 = \lambda$, and the maps $\mathcal{L}$ and $\mathcal{D}$ in Theorem 4 are given by holomorphic respectively meromorphic functions

$$x\colon \{t \in \mathbb{C} \mid t\bar{t} < 1\} \to \mathbb{C}$$

and

$$a\colon \{t \in \mathbb{C}^* \mid t\bar{t} < 1\} \to \mathbb{C},$$

where a has a first order pole at $t = 0$ and is holomorphic elsewhere. These functions can be approximated by their Taylor respectively Laurent series. Note that both functions are odd in t. Moreover, due to a symmetry of the Lawson surface covering $z \mapsto iz$ which is not space orientation preserving, see [9], the series coefficients x_k of the function x vanish if $k \mod 4 \neq 1$ and the series coefficients a_k of the function a vanish if $k \mod 4 \neq 3$. Moreover, the coefficients x_k are real multiples of $\frac{1+i}{4}$ and the coefficients a_k are real multiples of $\frac{1-i}{4}$ due to a anti-holomorphic symmetry of the Lawson surface covering $z \mapsto \bar{z}$.

The numerical search for the coefficients of x and a has been designed as follows: We have implemented the extrinsic closing condition from the beginning and searched for a finite number N of real coefficients of the numerical approximates x^N and a^N :

$$x^N(t) := \frac{1+i}{4}((1 - x_1 - x_2 - ... - x_N)t + x_1 t^5 + x_2 t^9 + .. + x_N t^{4N+1})$$

and

$$a^N(t) := \frac{1-i}{4}((1 - a_1 - a_2 - ... - a_N)\frac{1}{t} + a_1 t^3 + a_2 t^7 + .. + a_N t^{4N-1}).$$

Then we have chosen a finitely many $K >> 2N$ sample points t_k in equidistance on an arc with angle $\frac{\pi}{2}$ on the circle. Note that a quarter of the circle is enough due to the symmetries of the Lawson surface and of the functions. Then we numerically minimized the functional

$$\mathcal{F}\colon \mathbb{R}^{2N} \to \mathbb{R}; (x_1, .., x_N, a_1, ..a_N) \mapsto \sum_{k=1}^{K} \parallel a^u(x^n(t_k)) - a^n(t_k) \parallel^2$$

with the help of the *FindMinimum* routine in *Mathematica*. For example for $N = 10$ and $K = 120$ we have found a numerical root of this functional with an error of 10^{-12} which seems reasonable good compared with the expertise of earlier experiments on k-noids by the second author. The image of the unit circle of these functions is shown in Figure 3. Note that the (numerical computed) surface obtained out of the

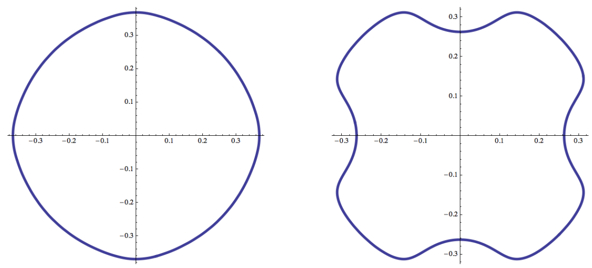

FIGURE 3. The spectral data (the complex anti-linear and the complex linear part of the flat connections on the eigenline bundles) of the Lawson genus 2 surface along the unit circle of the spectral plane. These data completely determine the associated family of flat connections of the Lawson surface $\xi_{2,1}$.

spectral data by means of Theorem 2 has the symmetries φ_2, φ_3 and τ. Moreover, it has the additional space orientation reversing and the anti-holomorphic symmetries discussed above. From this one can deduce that the so constructed minimal surface in S^3 must be the Lawson surface. In fact, the energy formula in [10] applied to our numerical spectral data yields an area of 21.91, a value which only slightly differs from what has been numerically computed in [13] using the Willmore flow.

The reconstruction of CMC surfaces as in Theorem 2 has been implemented in the software suite *Xlab* by the second author. However, the input data must be given as a DPW potential. Therefore, we applied the transformation (2.8) to obtain a DPW potential of the form (2.7) for $z_0 = 1$ and $z_1 = i$. Note that the functions $A(x(t), a(t))$ and $B(x(t), a(t))$ are automatically even in t which means that we have obtained holomorphic functions $A(\lambda)$ and $B(\lambda)$ depending on the spectral parameter $\lambda \in \{\lambda \in \mathbb{C} \mid \lambda\bar{\lambda} < 1 + \epsilon\}$. The symmetries imply that A and B have real coefficients and that they are also even with respect to λ. An image of the Lawson surface of genus 2 is shown in Figure 4. Note that the existence of such an image also serves as a positive test for our numerical experiments.

4. EXPERIMENTS: WHITHAM DEFORMATION OF LAWSON SYMMETRIC CMC SURFACES OF GENUS 2

The physical idea behind these experiments is the following: Starting with the Lawson surface of genus 2 and changing the pressure inside the Lawson surface slightly will make compact CMC surfaces in S^3. As these small deformations should be unique by physical reasoning the CMC surfaces should again be Lawson symmetric, so we can use the DPW potential in (2.7) to construct them. The main difference to the Lawson surface is that there are no space orientation reversing symmetries anymore as the pressure inside and outside the CMC surface differs. Therefore, the functions $A(\lambda)$ and $B(\lambda)$ are not even anymore. This can also be deduced from the Sym point condition (2.9) and (2.10).

As we have discussed in section 2 there is a complex one-dimensional family of Riemann surfaces of genus 2 which admit the holomorphic Lawson symmetries. But the

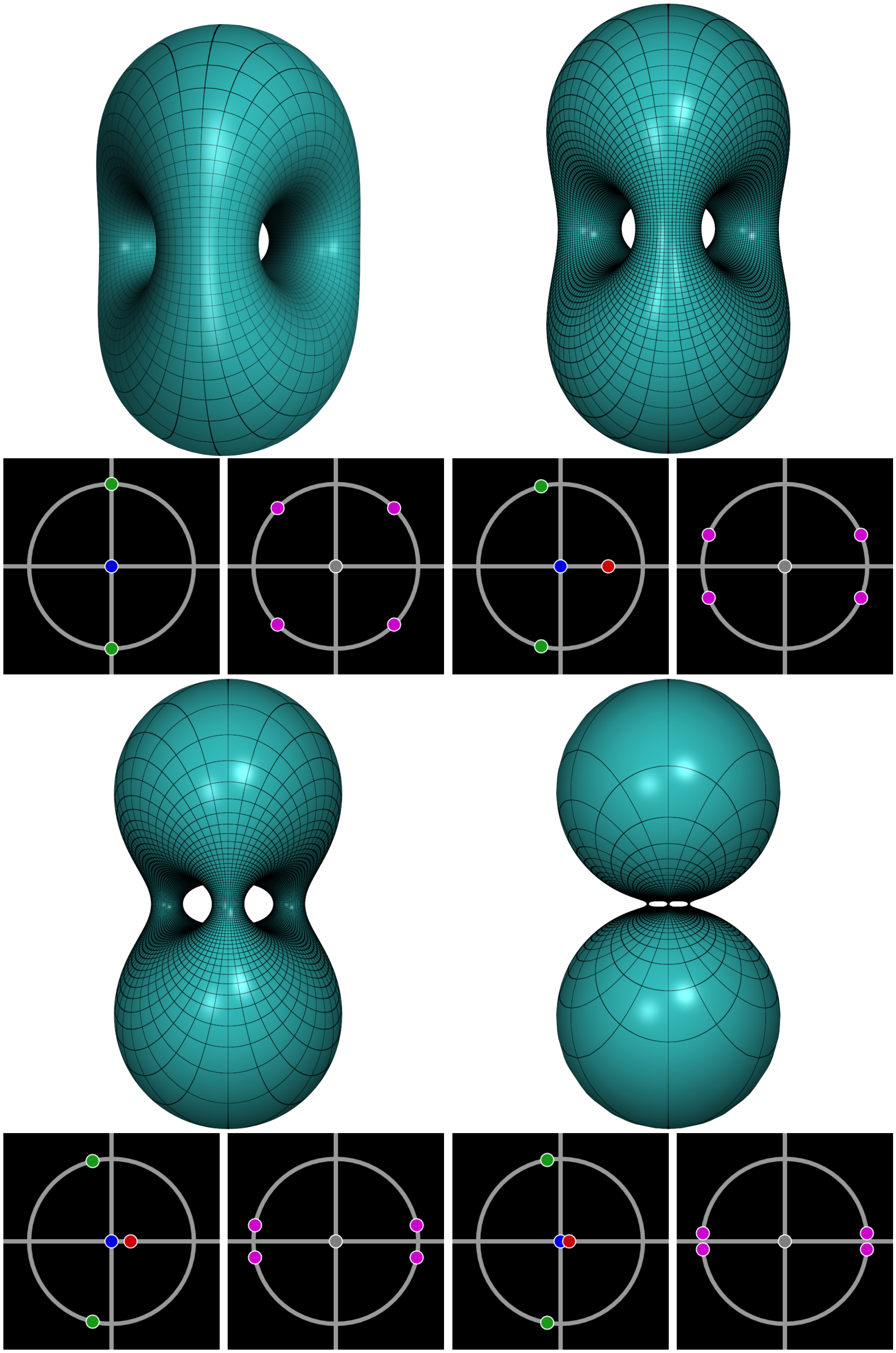

Figure 4. A family of CMC surfaces of genus 2, starting with the Lawson surface in the upper left corner, together with their spectral curves and their Riemann surface type.

physical insights only indicates a real one-dimensional family of Lawson symmetric CMC surfaces. By an analogy to tori we may expect that the real one-dimensional family of Riemann surfaces induced by Lawson symmetric CMC surfaces consists of those surfaces given by (2.6) with $\bar{z}_0 = z_1$, $z_0\bar{z}_0 = 1$, which we call rectangular Lawson symmetric surfaces from now on. Our experiments suggest that this is true, see Figure 4. In fact, we have not found any Lawson symmetric CMC surface whose Riemann surface is not rectangular.

We have designed our experiments as follows: Start with a rectangular Lawson symmetric Riemann surface and with the corresponding DPW potential (2.7). Write

$$A = \sum_{k=0}^{\infty} a_k\lambda^k \text{ and } B = \sum_{k=0}^{\infty} a_k\lambda^k$$

and approximate them by

$$A^n = \sum_{k=0}^{N} a_k\lambda^k \text{ and } B^n = \sum_{k=0}^{N} a_k\lambda^k.$$

Define the functional

$$\mathcal{F}_1\colon S^1 \times \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{R};\ (\lambda, a_1, .., a_N, b_1, .., b_N) \mapsto \sum(\operatorname{Im} t_{i,j})^2 + \sum(\chi(\operatorname{Re} t_{i,j}))^2$$

where $t_{i,j} = \frac{1}{2}\operatorname{tr}(M_iM_j)$ for the monodromy matrices M_i of the connection

$$\nabla = d + \eta_{A(\lambda),B(\lambda)}$$

on the four-punctured sphere $\mathbb{CP}^1 \setminus \{\pm z_0, \pm z_1\}$ and

$$\chi\colon \mathbb{R} \to \mathbb{R}; x \mapsto \begin{cases} 0 & x \in [-1, 1] \\ \parallel x \parallel & \text{otherwise} \end{cases}.$$

Next, we impose the extrinsic closing conditions (2.9) and (2.10) in our search: Write B as

$$B = fR + hC \tag{4.1}$$

where f is the unique polynomial of degree ≤ 3 satisfying

$$f(\lambda_1) = z_0^2\lambda_1,\ f(\lambda_2) = z_1^2\lambda_2,\ f'(\lambda_1) = z_0^2,\ f'(\lambda_2) = z_1^2$$

and

$$h(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_2^2)$$

for the Sym points $\lambda_1, \lambda_2 \in S^1$. We again approximate

$$C = \sum_{k=1}^{N-4} c_k\lambda^k.$$

There is no reason to assume that the anti-holomorphic symmetry of the Lawson surface is broken for the rectangular Lawson symmetric CMC surfaces. Therefore, after rotating the spectral plane such that $\bar{\lambda}_1 = \lambda_2$, we work with the assumption that A and B are real, i.e., the coefficients a_k, c_k are real numbers. We fix $\bar{\lambda}_1 = \lambda_2$ and

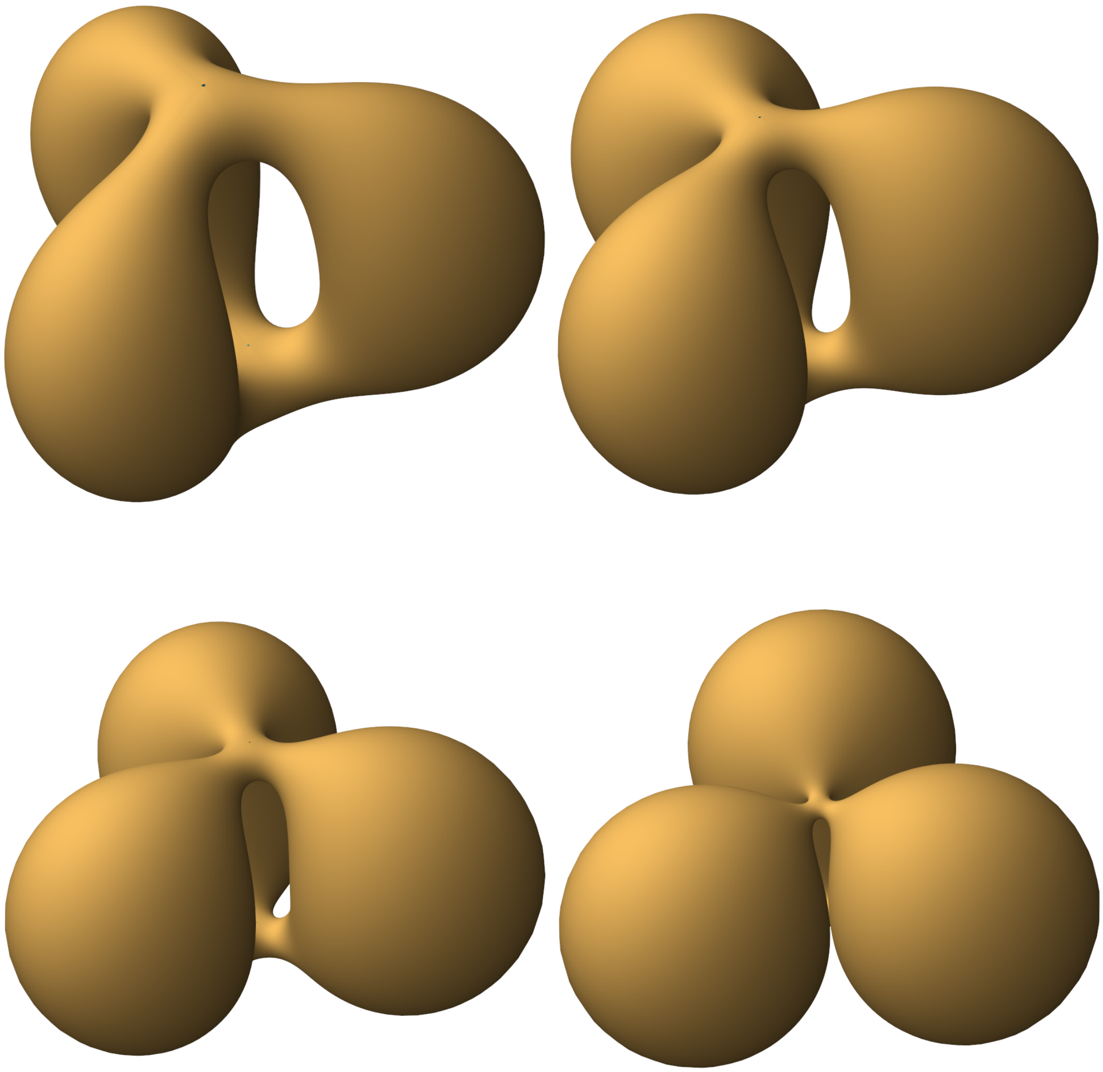

FIGURE 5. Unlike the CMC Lawson family in Figure 4, this family of genus two CMC surfaces in the 3-sphere is not connected to Lawson's minimal surface $\xi_{2,1}$. The family is conjectured to limit to a necklace of three CMC spheres as the conformal type degenerates (lower right).

define a functional as follows: Take a finite number of sample points $\lambda_3, .., \lambda_K \in S^1$ in equidistance and define

$$\mathcal{F}\colon \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R};\ (\lambda_1, a_1, .., a_N, c_1, .., c_N) \mapsto \sum_{k=3}^{K} \mathcal{F}_1(\lambda_k, a_1, .., b_N).$$

where the b_k are computed according to (4.1). Then, we searched numerically for minimizers of $\mathcal{F}$ starting with the initial data of the Lawson surface on a slightly deformed rectangular Lawson symmetric Riemann surface. We found evidence, i.e., the

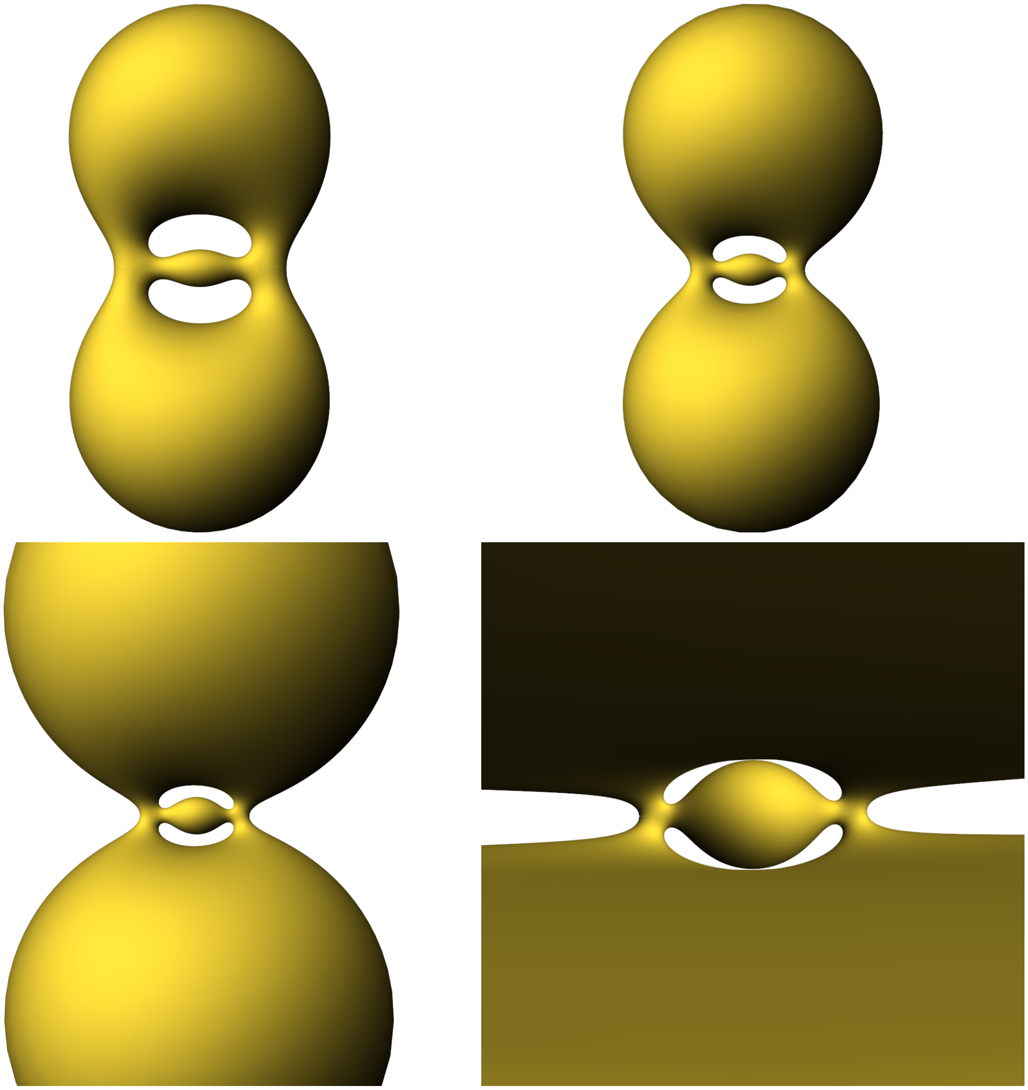

FIGURE 6. The Delaunay perspective of the surfaces in Figure 5.

numerical search has found a minimum of order 10^{-12}, for the existence of a Lawson symmetric CMC surface nearby the Lawson surface. We have repeated this method in order to obtain a family of Lawson symmetric CMC surfaces through the Lawson surface itself. The family of Lawson symmetric CMC surfaces is shown in Figure 4 together with images of their DPW spectral curve and with the corresponding four-punctured sphere defining the Riemann surface structure. The meaning of the DPW spectral curve picture is as follows: The circle is the unit circle in the spectral plane, whereas the green points are the Sym points. The blue point is just the center $\lambda = 0$ of the spectral plane, while the red point coming inside the unit disc is a zero of the function B.

4.1. **Apparent singularities in the DPW potential.** Looking at the DPW potential (2.7) more carefully, one observes that a zero of B causes a pole in the upper right corner of $\eta_{A,B}$. For a DPW potential corresponding to a CMC immersion, this pole must be apparent in λ. By the reality condition, this holds automatically if all zeros of B are outside the unit disc. When a (simple) zero λ_0 of B is inside we have to ensure that the pole is apparent. There are in principal four possibilities (in the genus 2 case): $A(\lambda_0) = -\frac{2}{3}$ and $A(\lambda_0) = \frac{1}{3}$, so that the singularity in the upper right corner is removable, or $A(\lambda_0) = -1$ and $A(\lambda_0) = 0$, so that the lower left corner of the DPW potential has a zero at λ_0 and the first order pole in the upper right gets apparent by a diagonal gauge only depending on λ.

The family of CMC surfaces through the Lawson surface corresponds to the case $A(\lambda_0) = -\frac{2}{3}$ where λ_0 is the zero of B of smallest distance to $\lambda = 0$. The family of Lawson symmetric CMC surfaces converges against a doubly covered minimal sphere while the zero of B converges to $\lambda = 0$. The reason for this is that the Hopf differential is given by the pull-back of $B(0)\frac{dz^2}{(z^2-z_0^2)(z^2-z_1^2)}$ and hence vanishes in the limit, so the corresponding surface is totally umbilic and therefore a covering of the round sphere. Continuing this family through the double covered sphere produces the same Lawson symmetric CMC surfaces, but this time inside out, until we end at the Lawson surface again. This gives one component of the space of Lawson symmetric CMC surfaces in S^3, see Figure 7.

There also exits a distinct family of rectangular Lawson symmetric CMC surfaces. To find this family, we have implemented the condition that at the zero λ_0 of B inside the unit disc we have $A(\lambda_0) = -1$, and apart from that we have used the same methods as above. This family converges against a chain of three round CMC spheres in S^3, see Figure 5. In Figure 6 we show the same surfaces but this time after the stereographic projection preserving the symmetries φ_2 and τ. Basically, they are almost 2-lobed Delaunay tori where a piece of a Delaunay cylinder is glued in.

We have not been able to find any surfaces for the remaining two cases $A(\lambda_0) = 0$ and $A(\lambda_0) = \frac{1}{3}$. If such families would exists, they could not converge against a chain of spheres. This follows from the energy formula $E(f) = -12\pi A(0)$, which implies $A(0)$ must be negative. Moreover, we have not found any Lawson symmetric CMC surfaces which are not rectangular. There is no reason to believe that they could not exists. Nevertheless one could expect that they are only immersed not embedded, thus only exists at a "higher energy level".

5. Experiments: The Lawson surfaces $\xi_{g,1}$

A natural generalization of our experiments is given by looking at deformations of the Lawson surfaces $\xi_{g,1}$ of genus g. These are quite similar to the Lawson surface of genus 2 but now have a $g+1$-fold symmetry instead of the threefold one. By analogy, we used the following DPW potential

$$\eta = \eta_{A,B} = \pi^* \begin{pmatrix} -\frac{g}{g+1}\frac{z(2z^2-z_0^2-z_1^2)}{(z^2-z_0^2)(z^2-z_1^2)} + \frac{A}{z} & \lambda^{-1} - \frac{(A+\frac{2}{g+1})(A+\frac{1-g}{1+g})}{B} z^2 \\ \frac{B}{(z^2-z_0^2)(z^2-z_1^2)} - \frac{\lambda A(A+1) z_0^2 z_1^2}{z^2(z^2-z_0^2)(z^2-z_1^2)} & \frac{g}{g+1}\frac{z(2z^2-z_0^2-z_1^2)}{(z^2-z_0^2)(z^2-z_1^2)} - \frac{A}{z} \end{pmatrix} dz \tag{5.1}$$

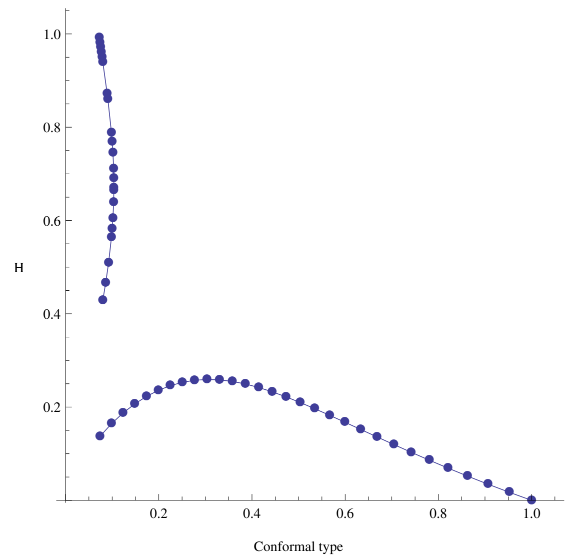

FIGURE 7. This graph represents two families of genus two CMC surfaces based on Lawson's minimal surface $\xi_{2,1}$ in the 3-sphere, plotting conformal type against mean curvature. The CMC Lawson family starts at Lawson's surface at the lower right and limits to a doubly covered minimal 2-sphere at the origin at the lower left (see Figure 4). The plot at the upper left represents a separate family conjectured to limit to a necklace of three CMC spheres as the conformal type degenerates (see Figure 5).

in order to perform experiments for Lawson symmetric CMC surfaces of genus g with Riemann surface structure given by $y^{g+1} = \frac{z^2-z_0^2}{z^2-z_1^2}$. The Sym point condition is almost the same as in the case $g = 2$ with the only difference that the function R in (2.10) is given by $R = A(A + \frac{1-g}{1+g})$. We have performed the experiments for $g = 1, .., 8$ totally analogous to the case of $g = 2$. In all cases we have obtained for $z_0 = 1$, $z_1 = i$ the Lawson surface $\xi_{g,1}$, see Figure 1, and for small rectangular variations of the Riemann surface structure we have obtained CMC deformations through Lawson symmetric surfaces. We thus have found numerical evidence that for all genera g there exists Lawson symmetric CMC deformations of the Lawson surface $\xi_{g,1}$. Especially, in the case of $g = 1$ we have recomputed the Clifford torus and CMC deformations of it which are of course the homogeneous tori of spectral genus 0 first and then bifurcate to the Delaunay tori of spectral genus 1. The bifurcation can be explained in our setup as follows. The zero $\lambda_0 > 1$ of B next to the origin is of order 2 for the Clifford torus. When it crosses the unit circle it can continue either as a double zero to the inside or bifurcate to two simple zeros reflected across the unit circle. When it continues as a double zero the CMC tori remain homogenous whereas in the second case one obtains unduloidal rotational Delaunay tori of spectral genus 1. We have done the corresponding experiments for these tori in the DPW approach independent to the well-established theory of spectral curves for CMC tori. We like to mention that the numerics worked in that case as good as in the case of genus $g \geq 2$ surfaces giving again evidence for our experimental setup.

6. Computational aspects

The surfaces were computed using XLab, a computer framework for surface theory, experimentation and visualization written in C++. XLab implements the DPW construction [5] of CMC surfaces in $\mathbb{S}^3$ in three steps:

- The holomorphic frame is computed as the numerical solution to an ordinary differential equation. Loops appearing in the DPW construction are infinite dimensional; for computation, they are represented finitely as Laurent polynomials about $\lambda = 0$ by chopping off the infinite Laurent series to heuristically determined powers, typically running from λ^{-40} to λ^{40}. This chopping is similar to the way real numbers are represented by rational numbers for numerical computation.
- The unitary frame is computed from the holomorphic frame via loop group Iwasawa factorization. This calculation applies linear methods to matrices of coefficients of the Laurent polynomials representing the holomorphic frame [17].
- The CMC immersion is computed by evaluating the unitary frame at the sym points.

The most difficult part of the construction of the CMC families was the search for the accessory parameters in the DPW potential for which its monodromy is unitarizable. As with the holomorphic frame, the infinite space of accessory parameters was made finite by chopping off its power series. The accessory parameters were compute by optimization (minimization) algorithms. The objective function for the search measured how far the monodromy of the DPW potential was from being simultaneously unitarizable. This measure was computed as the average over a set of equally spaced sample points on the unit circle in the λ-plane. To speed up these lengthy calculations, the objective function was computed in parallel over the sample points simultaneously. Once the accessory parameters in the DPW potential were found, the diagonal unitarizer, computed as in [19], was used as the initial value for the holomorphic frame.

Each Lawson CMC surface was built up by applying its symmetry group to one fundamental piece computed by the DPW construction. The completed surface was viewed, manipulated and rendered in the XLab $\mathbb{S}^3$ viewer.

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Sebastian Heller, Institut für Mathematik, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tubingen, Germany

Nicholas Schmitt, Institut für Mathematik, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tubingen, Germany

E-mail address: heller@mathematik.uni-tuebingen.de

E-mail address: schmitt@mathematik.uni-tuebingen.de