

Details and Mathematical Description

Cellular automata

We know that the earth behaves in a very complex way: every single, little, action interferes with the other in a large-scale, chaotic dynamic. This dynamic is very difficult to study, so, often, we assume that the nature behaves in a "local"-way: we study a phenomenon by considering only the surroundings and forgetting all about everything else.

In the fifties of the twentieth century, in order to investigate such "local" behavior, John von Neumann (1903-1957) and Stanislaw Ulam (1909-1984) developed a "cellular automaton", which is very useful to study some natural dynamics, with a wide range of medical applications and information technology.

Abstractly, a cellular automaton is constituted by a grid of "cells". Every cell has a state, chosen from a "discrete" set of possible cases, for example "on"-1 and "off"-0. Every cell has a neighborhood too: a set of cells that we will consider near the considered cell.

The evolution of the cellular automaton is described by some rules, that allow to get from the initial step - or step 0 - to the following

The cellular automaton presented here is formed by a single line of cells. The cell's state can be "0" or "1" and the neighbors of a cell are the one on the left, the one on the right and the cell itself too. The state of a cell depends on the state of its three neighbors in the previous step. So the rule is described giving the behaviour of the cell in the 8 different conditions of the neighbors. Evolution is represented by a grid in which each row corresponds to a step.

Tasks:

- When does the cellular automata produce symmetric patterns?
- Try to describe rule 4
- Try to understand how the rule number is calculated
- Try to describe rule 146

Questions and answers:

Q: What are the differences between the blue squares and the green squares in the rules?

A: For every figure, the green square is the considered cell, and the blue squares are the neighbours of this cell in the previous step. You click on the green square to decide whether the cell will be activated or not. You don't click on the blue squares because in the images there are all the possible configurations already.

Q: Are there any differences to choose the rules by using the slider or by clicking on the tetramins?

No. The slider is automatically set up when you click on the green squares as well the squares are when you move the slider.

Q: Is the final output influenced by the order I choose the initial state?

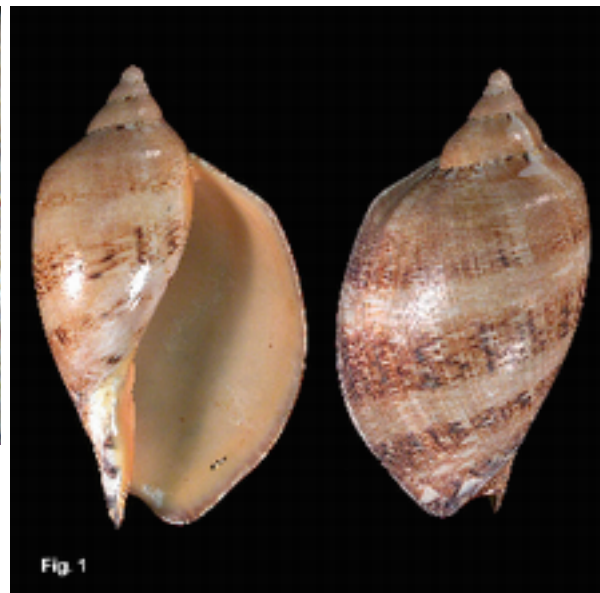
The final output does not depend on the order you click the cells. Of course different pattern give rise to different output.

Q: Is either the automata a line or a torus?

Is a line. Even though it could be interesting to study the “toric” version too.

Q: How are cellular automata related with nature? Is it possible to see that relation with the program?

A explicative example is provided by biology. The shells of Cymbiola genus exhibits a cellular automaton pattern that are very similar to the ones you can create with the program. Have a look to these picture and try to reproduce them with the automata





References:

- S. Wolfram: Los Alamos Science, 9 (Fall 1983) 2-21
<http://www.stephenwolfram.com/pdf/Cellular-Automata-Stephen-Wolfram-Article.pdf>
- S. Coombes: The Geometry and Pigmentation of Seashells
<http://www.maths.nott.ac.uk/personal/sc/pdfs/Seashells09.pdf>
- Image on the top-left: http://en.wikipedia.org/wiki/File:Textile_cone.JPG
- Other Images: <http://seashellsofnsw.org.au/>

Temperatures

Scientists argue daily with a problem: they collect large amounts of data and they need instruments that can process the data and a way to display them concisely to the general public.

Starting from the data given to us by the ARPA (which in italian stands for "Regional Agency for Environmental Protection") of the italian region of Lombardy we have realized these applications to help you to understand this problem and how mathematics can help to solve it. We have obtained the temperatures from 1990 to 2012 recorded hourly by a control unit in Milan. In total we have 192,840 data!

With the first application you can choose a time interval and check the temperature trend. You also get the highest and lowest temperature for the period you have chosen.

With the second application, you can use the basics of statistics by computing average, maximum and minimum on different time scales. You can also display the least square line, a line which has the property of assessing the data so that, globally, the actual values deviate the least possible from those of the line.

Tasks:

- Is it interesting to communicate the data calculated every six months in order to give a global idea of the trend of the temperatures? To answer this, chose to calculate the means every six months from February and after from November...
- In relation to this issue, how does the least squares line behaves?

Questions and answers:

Q: It seems that there are “holes” in the data. Why?

A: There are “holes” in fact and these correspond to invalid data caused by malfunctioning of the control unit.

Q: What happens if we calculate maximum temperatures, minimum and average every six months? Does the behavior of these values depends on when we start to calculate the data? Try to display the data for one or more years starting from February and after from November...

A: One can see different behavior because February is the month with the lowest (average) temperature of the year whereas November is between the hottest month and the coldest one.

Q: Does the same thing happens if we choose to calculate these numbers every year instead of every six months?

A: With everything calculated every year, the starting point doesn't interfere too much with the values because one year is a periodic interval for the temperatures whereas six months is half of a periodic interval.

Q: Where does the dataset come from?

The dataset used by the application are provided by the Regional Agency for Environmental Protection (in italian, ARPA) of Lombardy. It consist of the hourly temperatures obtained by a control unit in Milan from 1990 to 2012.

References:

- Sheldon M. Ross, Probability and Statistics for Engineers and Scientists, Academic Press, 2009
- ARPA Lombardia (Agenzia Regionale per la Protezione dell'Ambiente) - <http://ita.arpalombardia.it/ita/index.asp>

Dynamical Systems

Consider an ecosystem in which there are only two kinds of populations: prey and predators. The model is founded on three assumptions:

- 1) prey have a practically infinite available amount of food, which means that the speed of growth of the prey increases as the number of prey increases.
- 2) If there are more prey, it is easier for a predator to find them; if there are more predators, it is easier for a prey to be eaten. So the probability that a predator eats a prey increases as the product of the number of predators and the number of prey increases.
- 3) Predators are in competition one another, which means that the speed of growth of the predators decreases as the number of predator increases.

The model is described by two equations of the type:

$$x' = Ax - Bxy,$$

$$y' = Cxy - Dy,$$

where x represents the number of prey, y the number of predators and x' and y' the respective rates of growth. "A", "B", "C" and "D" are positive parameters that depend on the considered situation. "A" expresses the rate of reproduction of the prey, and is related to the assumption (1). "B" is linked to the killing rate of prey by predators, "C" is related to the rate of successful attacks by predators on prey. "B" and "C" are related to the assumption (2). "D" expresses the fact that, if the number of predators is very high, then they could not find enough food and then they could die of hunger and is related to the assumption (3).

Tasks:

- Is it possible to find stable dynamics? (it means that the number of animals is the same at every time)
- What is the behaviour of the system when $A=0$?
- What does it happen if you choose an initial point with negative coordinates? Is there any physical interpretation of this choice?
- What is the behaviour of the system when you hit the (0,0) point?

Questions and answers:

Q: Why does the speed of the dot representing the state of the system change? Does this speed depend on the initial data?

The speed of the dot is calculated by the dynamical system: (x', y') is the speed. So it is reasonable that speed changes. The speed depends on the position of the dot.

Q: It seems that the orbit gets large when evolving for a lot of times. Is it right?

That is a problem of the approximation of the computer. Every number has a precision of about 7 decimal digits, so after a while the error becomes big and the simulation is inaccurate.

References:

- E. R. Leigh, "The ecological role of Volterra's equations"
- Kaplan, Daniel, Glass, Leon, "Understanding Nonlinear Dynamics"

Wator

This interactive animation is inspired by the cellular automaton Wator. This automaton was created by the Canadian mathematician Alexander Keewatin Dewdney and presented in the article "Computer Recreations: Sharks and fish wage an ecological war on the toroidal planet Wa-Tor". The word "Wa-Tor" comes from the words "Water" and "Torus", which is the mathematical name for a 'donut' surface.

The differences between the original model and ours are needed to have more freedom in varying parameters governing the system.

Every animal occupies a cell of a rectangular grid and can move in the neighboring cells (chosen by the user) following some simple rules.

An animal of first species (fish), displayed in green color, moves randomly in an empty cell in the surroundings. After a given amount of time, it reproduces and generates another fish and, finally, dies.

An animal of the second species (sharks), displayed in red, eats randomly fish near to him; if there are no fish nearby, it moves randomly in an empty cell. After a given amount of time it give birth to another shark. If it cannot eat enough it dies.

Tasks:

- What happens if "Time birth sharks" is lower than "Time death shark"?
- Is it possible to recognize sharks from the fish without relying on color?
- Does the evolution of the system looks like the evolution of Lotka-Volterra?
- What is the difference if Random Initial Energy is set to 0% rather than 100%?

Questions and answers:

Q: Why the graph of the system is so irregular?

This program is a simulation, that means the animal moves randomly. This give raise to the irregularities. Using a large grid the irregularities should vanish.

Q: What means the different gradiation of colours?

The gradients shows the amount of energy of the animal. If a "new-born" fish is of a bright green and an "old" fish is of a dark green. The same holds for the sharks.

References:

- Dewdney (1984) "Sharks and fish Wage an ecological War on the toroidal planet Wa-Tor", Scientific American December.
http://home.cc.gatech.edu/biocs1/uploads/2/wator_dewdney.pdf

Cartography

A well-known riddle says: "the hunter saw a bear standing still 100 meters to the east, he moved north to 100 meters, shot straight south and struck the animal. What color was the bear?".

What's the catch? That's right, we are at North Pole, where the intuitive idea of cardinal points, usually associated with the concepts of "forward, backward, right, left" no longer works. At North Pole all directions are South and, vice versa, at South Pole all directions are North. This is a consequence of how we orient ourselves on planet Earth.

If we pay attention, we will notice a substantial difference between meridians, which follow the north-south direction, and parallels, which follow east-west one. If a plane were to connect two cities on the same meridian it would move on a route that would follow precisely the meridian connecting the cities, whereas if it were to connect two cities on the same parallel it will follow a completely different route (unless the parallel is the equator!).

And if we were to transcript these routes on a map? Will straight lines come up? How to draw a precise plane or naval route on a map of the planet Earth? What kind of map should be more suitable? Is there a "perfect map" which could represent Earth without distorting areas, angles or distances? And, in the real world, what is the shortest route to travel on, in order to connect two distinct points on the planet? What is the route to follow if we want to avoid bearing?

These problems have affected navigators, surveyors and mathematicians for centuries.

Six different projection are used.

-Plane chart: this projection represents both parallels and meridians with straight lines. Every line from one set is orthogonal to every line of the other set and they are both equally spaced on the map.

This map is obtained projecting every point on the Earth surface on a cylinder which has the same height as the globe and which is tangent along the equator and then cutting the cylinder along 180°E meridian and "unrolling" it onto a plane.

It is *not conformal*, *nor equal area*, but it *preserves distances along meridians*.

Thanks to the simple relation between the coordinates on the sphere and the ones on the plane it has become the standard projection, adopted e.g. by NASA.

-Mercator (85N-85S): this is probably the most known projection beside plane chart. As in plane chart parallels and meridians are represented by two set of orthogonal straight lines, but parallels are not equally spaced. Mercatore is a *conformal map*, i.e. angles are locally preserved. Also, loxodromes are represented by straight lines on the map. This map does not preserve area, creating big distortions at high latitudes. For example, Greenland is represented the same size than Australia, when in reality it is more than three times smaller.

-Miller projection: this is a compromise map: it gives away conformality in order to achieve a less distorted representation; actually it is a deformation of Mercator map, in order to

include polar regions. It is designed to give a Mercator-like look. It is designed to reduce some area distortion induced by Mercator map.

-Lambert cylindrical: this is an *equal area projection*. Even if the shape of the continents is not preserved, nor distance between point, the area of closed shapes, and thus of continents, is preserved. As the previous cases it is undistorted along the equator, but increasingly dilated at higher latitudes, with poles represented by straight lines.

-Sinusoidal: this is also known as *Mercator equal area projection*; it does not represent loxodromes as straight lines, as Mercator map does nor it preserves angles, but it *preserves distances along parallels*. This is a non-rectangular map, in fact lateral boundaries are sinusoids. The 180°E meridian is represented by the whole contour of the map.

-Lambert azimuthal: this is an *equal area azimuthal projection*. Azimuthal means that the projection is not made on a cylinder, but directly onto a plane. It is not conformal, nor it preserves distances. Neither parallels nor meridians are represented as straight lines. The 180°E meridian is the whole boundary circle. Distortions increase at southern latitudes.

Tasks:

- Why doesn't a parallel represent the shortest path between two points?
- How can you characterize the geodesic paths?
- Is the geodesic path unique?
- And the rhumb line?

Questions and answers:

Q: What are loxodromes and geodesics?

A: Loxodromes (also called "rhumb lines") are lines that intersect every meridian at the same angle.

Geodesics are lines that represent the minimal path between two given points. On a sphere they are great circles.

Q: Why can't I draw a line to the poles using Mercator map?

A: Mercator map is a conformal map that uses the arctangent function in order to represent latitude. This function diverges at latitudes of $\pm 90^\circ$, thus a Mercator map representing polar regions will be indefinitely high.

Q: Conformal and equal area maps seem to be important. Why can't a map be both?

A: This is a consequence of Gauss's Theorema Egregium (Latin for: remarkable theorem) which states that the curvature of a surface is an invariant under local isometries.

References:

- Flattening the Earth: Two Thousand Years of Map Projections, John P. Snyder, 1993.
- Maling, Derek Hylton (1992), Coordinate Systems and Map Projections (second ed.), Pergamon Press.
- Monmonier, Mark (2004), Rhumb Lines and Map Wars: A Social History of the Mercator Projection (Hardcover ed.), Chicago: The University of Chicago Press.
- Snyder, J. P. [Map Projections--A Working Manual](#). U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office 1987.