Quasiperiodic Cartwheel Algorithm for *HB* Tiling Design

Uli Gaenshirt[1] and Amrita Acharyya[2]

[1]Sculptor & Researcher, Nuremberg, Germany; uli.gaenshirt@gmail.com
[2]Department of Mathematics and Statistics, University of Toledo, Ohio 43606, USA; amrita.acharyya@utoledo.edu

Abstract

In this paper, we propose an algorithm for creating figurative quasiperiodic tilings by duplicating individually drawn lines and assembling them according to the order of a quasiperiodic *HB* cartwheel. The Penrose *HB* tiling, first introduced in 2015 by one of us (*Uli*), is a variant of the Penrose rhombus tiling. The transformations of the two prototiles, elongated hexagonal tiles *H* and paper boat-shaped tiles *B*, are expressed as complex equations. We use representative examples for each step. The equations are illustrated by geometric and figurative images. Our goal is to work with computer scientists to develop a computer program based on the proposed algorithm. This would enable users to engage in an artistic design process. The program would visualize the edges drawn by the users in a large quasiperiodic *HB* cartwheel arrangement.

Introduction

In terms of the *quasiperiodic Penrose tiles* as decorated puzzle pieces, the largely unknown *Penrose HB tiling* (*H* stands for *hexagon* and *B* for *boat*) offers advantages over the well-known Penrose tilings P1, P2 and P3 [9][2], as its geometric prototiles have no acute 36° angles like these. The angles of the HB tiles are always multiples of 72°. The mirrored *H** and *B** tiles, which are required in the *HB* tiling, give the design a sense of balance. In 2025, we have already proposed some mirrored tiles in [2] using the kite and dart tiling P2, the Penrose rhombus tiling P3 and the *HB* tiling.

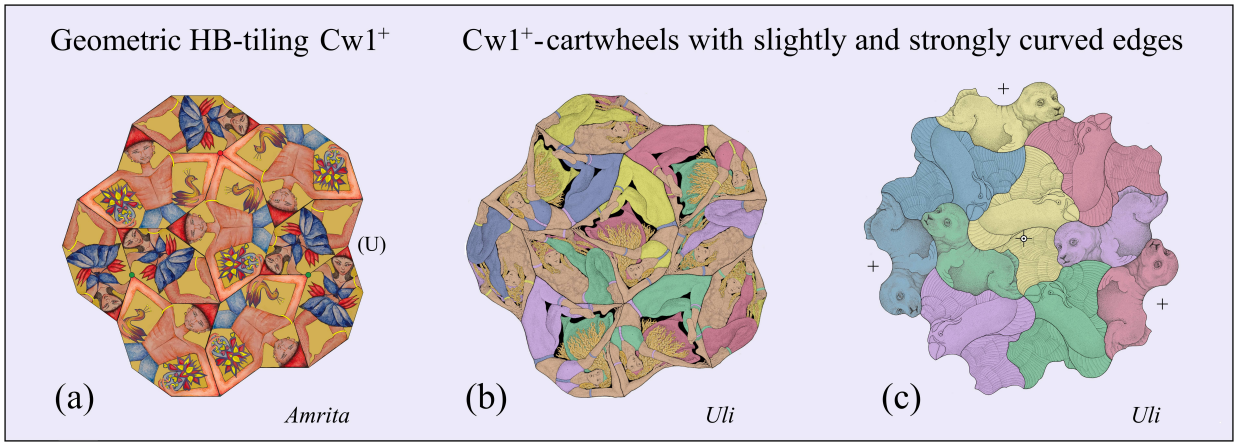

Figure 1: *Cartwheels Cw1*$^+$*: (a) Circus artists. (b) Ballet dancers. (c) Birds and baby seals.*

The geometric *HB* cartwheel *Cw*1$^+$ in Figure 1(a) is an arrangement of three *B* tiles, two *B** tiles, four *H* tiles, and one *H** tile. It has an asymmetrical center but an outline with *fivefold rotational symmetry*. When generating larger cartwheels [4], the numbers of the five different tile orientations become more and more similar. The result is a *structural, fivefold rotational symmetry* that cannot be *periodical*! The *curved edges* [5][6] in Figure 1(b) and 1(c) represent the *matching rules*, which prevent periodicity and enforce a *quasiperiodic* (*almost periodic*) *order*, even when the tiles are freely arranged. For the geometric *HB* tiles in Figure 1(a), the structural fivefold symmetry can only be achieved by marking the edges.

Matching Rules of Geometric *HB* Tilings Enforced by Triple Arrows

The geometric *HB* tiles in Figure 2(a) are provided with edge marks that represent the matching rules and create a quasiperiodic order with a structural fivefold rotational symmetry. We here propose *triple arrows*, which consist of three triangles arranged in a row. In the vertical *B* tile orientation, the triangle arrows always point upwards from the apex *V* to the top point *T*. For an edge *a*, the back triangle is black, for an edge *b,* the middle triangle is black, and for an edge *c,* the front triangle is black. At the "mirrored" edges *a**, *b** and *c**, the black triangles turn white and the white triangles turn black. Only edge types in Figure 2(c) have true mirror images. The arrows at the points *T* indicate the orientation of the tiles. In globally designed arrangements, the *B* tiles are the controlled tiles, while the *H* tiles merely fill the gaps.

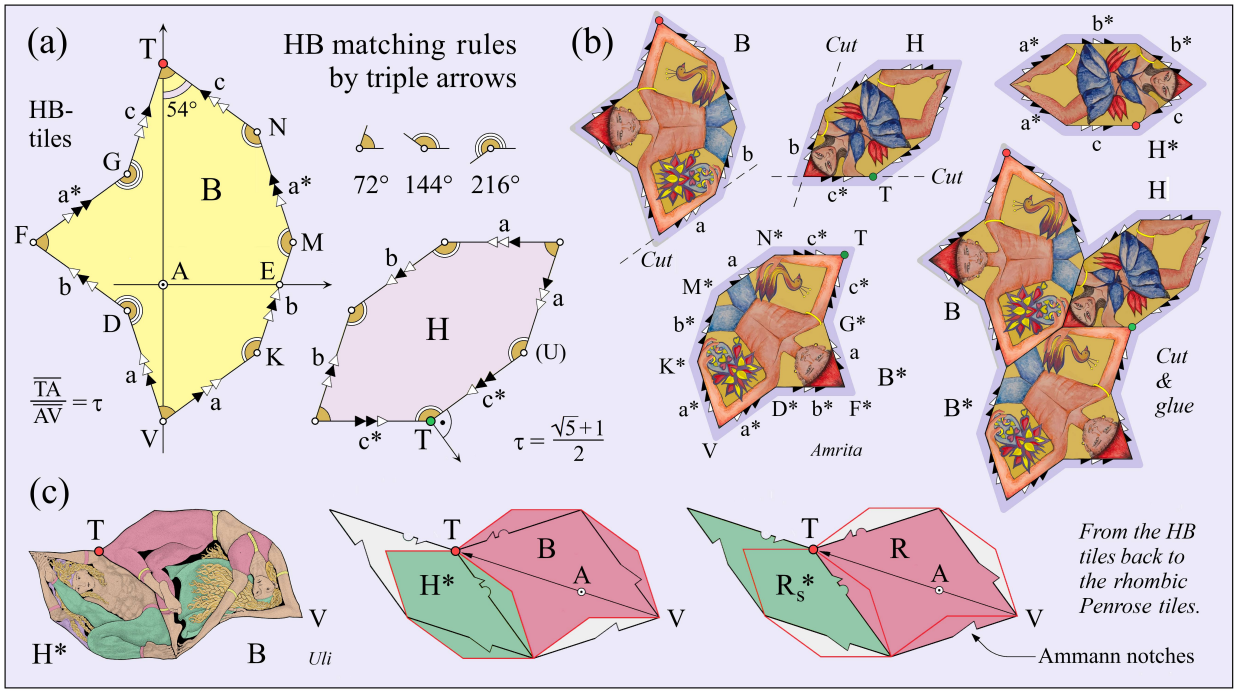

Figure 2: *(a) B and H tile. (b) Artists with edge marks. (c) Re-conversion of HB tiles into rhombs.*

In Figure 2(b), we propose a simple method for gluing tiles quasiperiodically together from paper copies. To do this, a border is left around the geometric tiles on which the outer half of the edge marks are drawn. The copies are then placed alongside each other so that identical marks are placed opposite. This is shown in Figure 2(b) on the left. One of the two opposite marks is then cut off along the dotted edge line. Now the three cut edges can be glued onto the uncut opposite borders, as shown on the right. To avoid *dead ends*, you should only glue tiles for which you have already designed a larger surrounding area!

The Idea of an Algorithm for Designing Quasiperiodic *HB* Cartwheel Tilings

Since *Amrita* no longer had time to continue the project in its originally planned form, the author (*Uli*) preferred to continue the work by sketching an algorithm for a computer program in the complex plane $\mathbb{C}$ that allows even inexperienced users to create artistic, quasiperiodic *HB* tessellations.

A cartwheel *Cw*4 with 180 tiles serves as a quasiperiodic framework for the algorithm. For clarity, we return to the Penrose rhombus tiling. Figure 2(c) shows the equivalence relation between the *HB* tiling and the rhombus tiling. The *Ammann notches*, which here define the matching rules, enforce mirror-image rhombs. Their indentations and protrusions correspond to the concave and convex angled edge pairs of the *HB* tiles. To avoid unnecessary detail, we will mark the edges in the following with the usual red Ammann line segments, even though the reflections of the rhombs are not visible in that way.

Arithmetical Description of the Neighborhood Transformations *h* in the Complex Plane C

The basis for the description of a Penrose rhombus tiling are the ten *neighborhood transformations h* [8]. In Figure 3, only h_1, h_2*, and h_3 are described in detail, as these can be considered as fundamental types. All ten valid neighborhood transformations prohibit periodic arrangements. In the following we calculate various, in some cases far-reaching transformations in the *complex plane* $\mathbb{C}$. The center of each rotation must be located at the origin *O* of the complex coordinate system!

The transformation h_1 in Figure 3(a) is, geometrically seen, a counterclockwise 72° rotation of a copy of the purple rhombus R_{id} around its top point *T*. However, according to the cartwheel structure, we assign the point A_{id} of R_{id} to the origin *O*. The point A_{id} divides the long rhombus diagonal $\overline{TV}$ of R_{id} in the golden ratio $\tau : 1$, with $\tau = (\sqrt{5}+1)/2$. The index notation *id* of R_{id} refers to the *zero identity* of A_{id} at the origin *O*. The *transformation* h_1, described in $\mathbb{C}$ as a rotation of R_{id} followed by a shift, generates one of the four *satellites* of the cartwheel *Cw*1. The rotation of the numbers *z* of the rhombus R_{id} with the angle φ_4 is written as the complex product $z \cdot w_4$ (The circular arrow φ_4 shows the rotation of z_T). The shift with the distance *l*, with $l = \tau / \cos 18°$, is expressed by the addition of $l \cdot v_2$. The index 2 denotes that the shift direction of *l* (yellow arrow) forms an angle of 36° counterclockwise to the real axis *Re*.

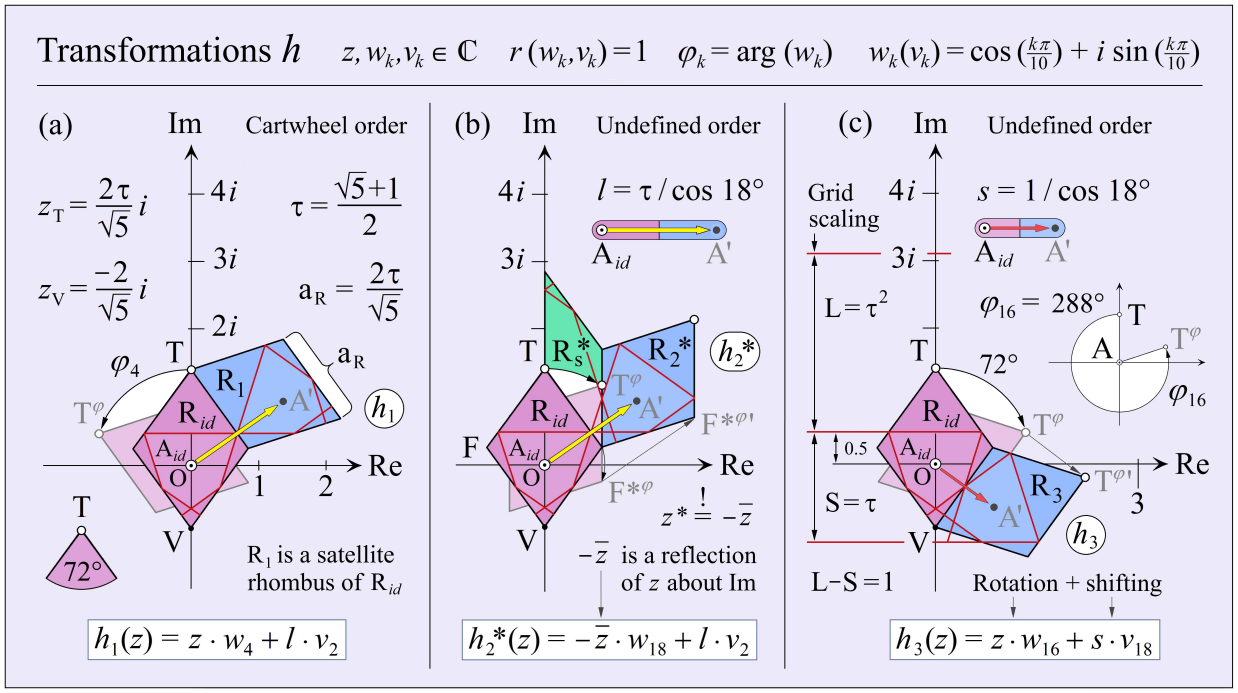

Figure 3: *Three of the ten neighborhood transformations h: (a) Satellite rhombus h_1 (R_{id}). (b) h_2*. (c) h_3.*

The transformation h_2* in Figure 3(b) differs significantly from h_1, because the rotation angle is only 36°. The gap between the thick rhombs is filled with a green skinny rhombus R_s which has acute angles of 36°. At the positions of the points *F* and $F^{*\varphi}$ can be seen that the rhombus R_{id} is first mirrored along its long diagonal before being rotated 36° clockwise. This is important for later use in the HB tiling. The shift of $F^{*\varphi}$ to $F^{*\varphi'}$ corresponds to an addition of $l \cdot v_2$. The result is the mirrored blue rhombus R_2*.

The transformation h_3 in Figure 3(c) is, geometrically seen, a clockwise 72° rotation over the point *V*. Expressed as a complex equation, the rotation is a multiplication of the numbers *z* and w_{16} (φ_{16} = 288°), i.e., the light purple rhombus in the background of R_{id} is rotated in the sense of the arrow 72° clockwise! The shift is the product of *s* and v_{18}, with $s = 1/\cos 18°$. The direction is 36° clockwise to the real axis!

Only the position of the transformation h_1 in Figure 3(a) corresponds to that one in a cartwheel *Cw*1, when this is centered as a basic *cluster* of seven rhombs in the origin of the complex plane $\mathbb{C}$.

The Grid Basis of the Cartwheel *Cw* 4 and its Hierarchical and Symmetric Structure

This version of the *quasiperiodic HB cartwheel algorithm* is limited to the size of the cartwheel *Cw*4 in Figure 4, which is created by a *quasiperiodic Ammann grid* [2, 3, 4]. The eight *Ammann bars* L and S are generated by the substitution $L_q \rightarrow LSLSL$ and $S_q \rightarrow LSL$. The sequence LSLSLLSL is called a *Fibonacci chain* or a *1D-Ammann grid*. The thickness of an L bar is τ^2 and that of an S bar is τ. These grid values determine the basic scaling. Five 1D-grids are superimposed in five not symmetrical orientations! The result is a *5D-Ammann grid*, in which the Penrose rhombs can be clearly assigned to the grid meshes [2].

Before examining the overall structure of the cartwheel *Cw*4, we will first calculate the purple $Cw1_{id}$ at the center of the complex plane $\mathbb{C}$, shown separately in the lower left. The equations of the four purple *satellite* rhombs are given on the left and right. The transformation h_1 was already shown in Figure 3(a) as one of the four satellites. In the four satellite equations, the complex numbers z are defined as points of the central rhombus R_{id}. We use $h_{\bar{1}}$, $h_{\bar{2}}$, … [7] as a short index notation for h_1^{-1}, h_2^{-1}, … [8]!

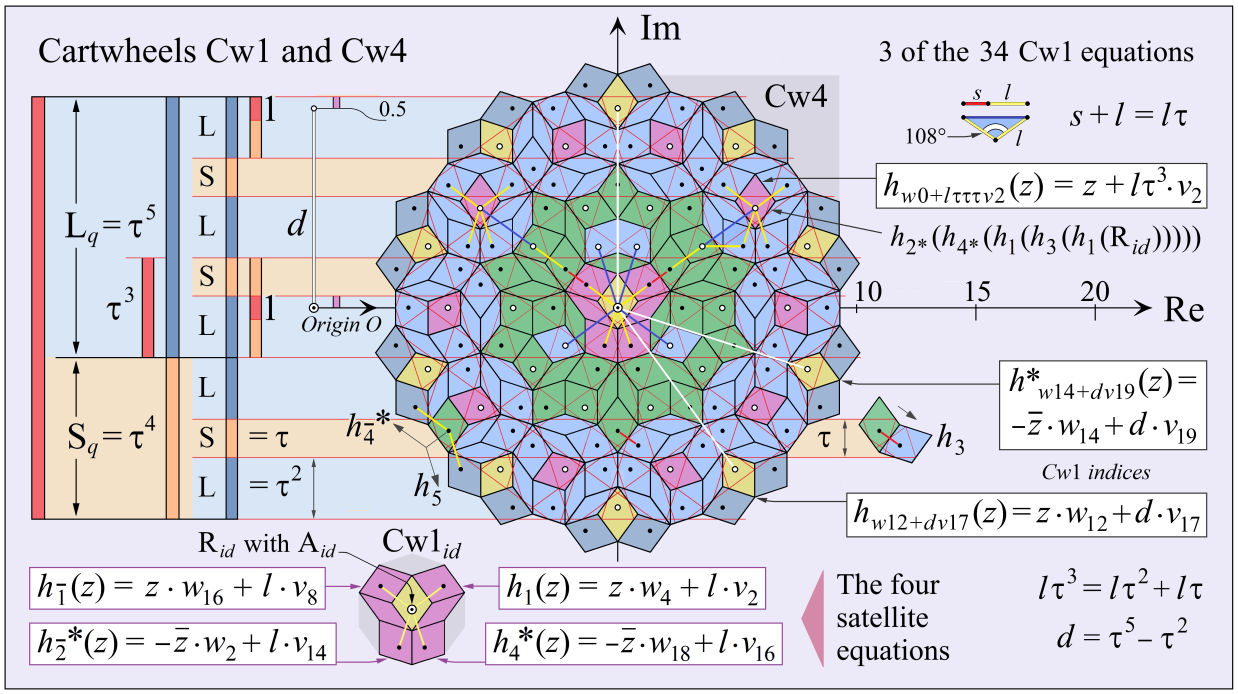

Figure 4: *Left: Grid. Middle: Cw4. Bottom: $Cw1_{id}$ with equations. Right: Full indexed Cw1 equations.*

Each rhombus *R* of a cartwheel *Cw*4 could be described by a path of successively executed neighborhood transformations *h,* starting from R_{id} [7]. However, the *Cw*4 cartwheel can also be created as a covering of 35 cartwheels *Cw*1 (or *Cw*1*) whose centers *A*° have only 20 distinct shift directions relative to the *Cw*4 center A_{id}. Therefore, an algorithm for modular transformations seems to be an effective approach. From the equation $l\tau^3 = l\tau^2 + l\tau$ we can deduce that the distance from the point A_{id} to the points *A*° of the outer purple rhombs is equal $l\tau^3$, because two yellow shift lengths *l* with an internal 108° angle can be replaced by a (blue) length $l\tau$. The index notation of $l\tau^3$ is $l\tau\tau\tau$. The three *Cw*1 equations on the right have the same form as the four satellite equations, except that *z* is defined here to be part of the overall $Cw1_{id}$.

The white points *A*° of the ten outer purple rhombs are arranged in a regular decagon. The ten points *A*° of the green rhombs also form a decagon, with the same directions from the origin, as well as the four points *A*° of the four inner light blue rhombs. In these shifts with the shift lengths $l\tau^3$, $l\tau^2$, and $l\tau$ the index numbers *k* of v_k are even, where $0 \le k \le 18$. In contrast, the shift directions with the lengths *d* to the points *A*° of the yellow rhombs, which are both decagonally arranged and regularly oriented, have v_k values with odd indices *k*, where $1 \le k \le 19$.

*Cw*4 Sector with Overlapping *HB* Cartwheels *Cw*1 and their Complex Equations

Figure 5 shows a sector of the cartwheel *Cw*4 as an *HB* tiling, which can be seen in the lower left corner as a reduced rhombus version. Inside the sector, there are five *B tiles and one B** tile with a white point *A*°, each surrounded by four *Cw*1 satellites. The five *B* tiles are congruent to the B_{id} tile without reflection, while the *B** tile and the cartwheel *Cw*1* are mirror images of B_{id} and $Cw1_{id}$. The two *Cw*1* satellites $h_{\bar{2}}^*(B^*)$ and $h_4^*(B^*)$, located next to the white *d* line and marked with red circles, are mirrored twice. Since two reflections cancel each other out, these two satellites of *B** are unmirrored images of B_{id}.

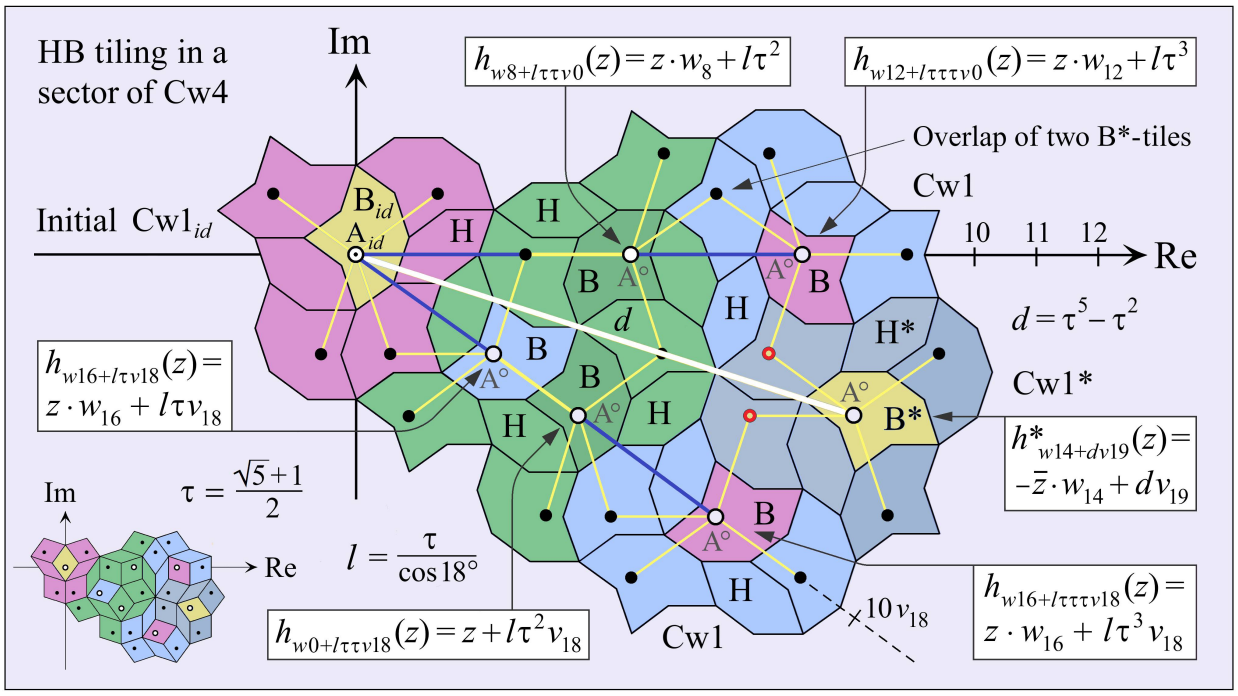

Figure 5: *Seven overlapping HB cartwheels Cw1 with six Cw1 transformation h related to* $Cw1_{id}$.

The gapless covering of the 36° sector by the seven *Cw*1 cartwheels serves as an example of how the 35 *Cw*1 cartwheels cover the entire *Cw*4 area. The six transformations *h* of the initial purple cartwheel $Cw1_{id}$ are representative examples of the 34 *Cw*1 transformation equations. The sector is half divided by the white *d* line that leads from A_{id} to *A*° of *B**. The three framed equations, that belong to the three *B* tiles below the *d* line, have the shift lengths $l\tau$, $l\tau^2$, and $l\tau^3$, where the common shift direction is the boundary of the sector. Above the *d* line, the sector is delimited by the positive axis *Re*, with only two points *A*° on it. In the two corresponding equations the number v_0 (= 1) is omitted because a factor 1 is unneccessary, just as w_0 (= 1) is omitted in the transformation $h_{w0+l\tau\tau v18}$. However, the numbers *w*0 and *v*0 are included in the indices to standardize them. Please note that we do not use indices within the indices!

With the standardized indices, the 34 *Cw*1 transformations can be easily applied to already computed results. As *z* is omitted in the indices, the mirroring transformations are identified by *h**, instead of $-\bar{z}$. Consequently, each index begins with the rotation number *wk,* followed by *+d* or *+l,* where *l* is always multiplied with τ, ττ, or ττττ. Finally, the shift direction is specified by a multiplication with *vk.*

The Transformations of an Edge e_{id} with Zero Identity to the Edges *a*, *b*, $a^{\bar{*}}$, and *c* of B_{id}

It's our aim to assign three hand-drawn curved edges *α*, *β*, and *γ* to the marked edges *a*, *b*, $a^{\bar{*}}$, and *c* of B_{id} and, in two further steps, to transform them onto the edges of the 115 *B* tiles of *Cw*4. Therefore, we have first to define a zero identity of an edge e_{id} inside $[A_{id}, E]$, with A_{id} in the origin. Then we have to define the transformations of $e_{id} \in \{a, b, c\}$ from $[A_{id}, E]$ to the corresponding edge marks of B_{id}.

In Figure 6(a), we show the yellow B_{id} tile in its standard position in the complex plane $\mathbb{C}$, with marked edges and the notations of its corner points. The generalized edge e_{id} is located on the real axis *Re* within the light green area of the interval $[A_{id}, E]$. The two separate intervals at the bottom right indicate that a reflection of e_{id} about the *Re* axis is written as $e^{\overline{*}}$ and that a reflection is visualized by turning the triangle color from white to black and vice versa. The choice of the edge type is the first step of the algorithm!

The interval $[A_{id}, G]$ results from a 108° rotation of $[A_{id}, E]$, counterclockwise around the origin, i.e., multiplied by w_6. $[A_{id}, G]$ contains two edge types, the type a on the left and $a^{\overline{*}}$ on the right, viewed in the direction of the triple arrows. There are three other doubled edge marks, multiplied with w_2, w_4, and w_8.

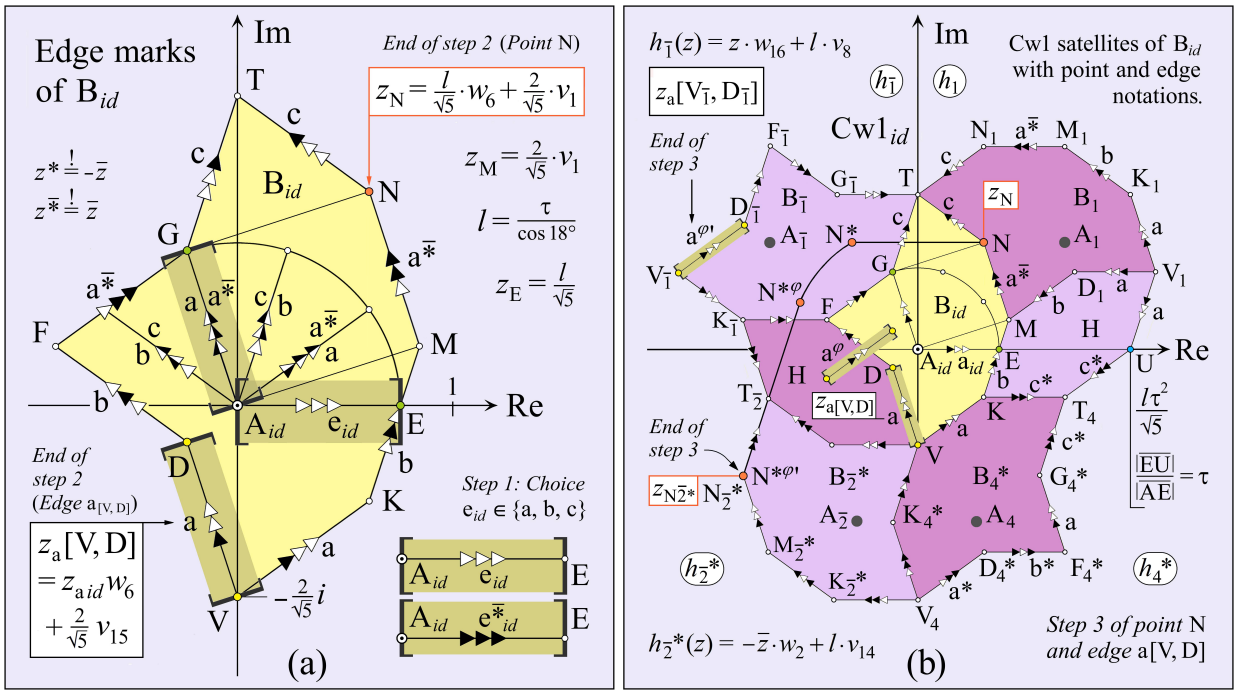

Figure 6: *(a) Transformations of edge marks. (b) Transformations, notations, and edge marks of $Cw1_{id}$.*

Example 1 of step 2 is the calculation of the number z_N. First $z_E = l/\sqrt{5}$ is rotated by a multiplication with w_6 to z_G. Then z_G is shifted by adding z_M, with $z_M = (2/\sqrt{5})\, v_1$. It results: $z_N = (l/\sqrt{5}) \cdot w_6 + (2/\sqrt{5}) \cdot v_1$. Example 2 of step 2 is the calculus of the lower left edge a of B_{id}. The numbers z_a of the interval $[A_{id}, E]$, short z_{aid}, are rotated by a multiplication with w_6, so that they are inside $[A_{id}, G]$. The numbers $z_a\,[A_{id}, G]$ are then shifted by the addition of z_V, with $z_V = (2/\sqrt{5})\, v_{15}$. It results: $z_a\,[V, D] = z_{aid} \cdot w_6 + (2/\sqrt{5}) \cdot v_{15}$.

Notations of the $Cw1_{id}$ Satellites and Illustrated Examples of both Steps 3 inside $Cw1_{id}$

Figure 6(b) shows the initial cartwheel $Cw1_{id}$ with the yellow B_{id} tile in its center, which is surrounded by four purple satellites. On the right half of $Cw1_{id}$ all edges and corner points of the satellites B_1 and B_4* are denoted. While in the cartwheel *Cw*4, the *H* tiles only fill the gaps between the dominant *B* tiles, in the separate illustration of the cartwheel $Cw1_{id}$ in Figure 6(b), an additional (blue) point *U* on the real axis *Re* with the real value $\tau^3/(\sqrt{5}\cos 18°)$ is needed to complete the outline of $Cw1_{id}$.

In the left half of $Cw1_{id}$, both results of the steps 2 in Figure 6(a) are continued by two steps 3. In the example for the point *N*, the transformation $h_{\bar{2}}$* to the lower left satellite is chosen. Consequently, z_N at the point *N* must first be mirrored to *N** and then rotated by φ_2 of w_2 to $N^{*\varphi}$. The shift by adding lv_{14} leads to $N^{*\varphi\prime}$, also labeled $N_{\bar{2}}$* or $z_{N\bar{2}*}$. In the example of the edge a, i.e., the point set $z_a\,[V, D]$, the transformation $h_{\bar{1}}$ to the upper left satellite is chosen. The edge a within $[V, D]$ is first rotated by φ_{16} of w_{16} to a^{φ}, and then shifted by adding lv_8. The result is $a^{\varphi\prime}$ or, more precisely, the set of the complex numbers $z_a\,[V_{\bar{1}}, D_{\bar{1}}]$.

The Construction of the Curved Edges and the Steps 1, 2, 3, and 4 of the Algorithm

Figure 7 shows that it is possible to substitute the straight edges *a*, *b*, and *c* with curved edges, the *open Jordan curves* (without self-intersection) α, β, and γ. This substitution can have a figurative, ornamental, or geometric character. In the lower center of the image, a schematic Im-Re diagram shows how a freely drawn curved edge α can be approximated by very short edge segments, which are generated successively by very small circles. The intersection of a circle with the curve α generates the center point of the next circle. The created points, emphasized in black, can be understood as complex numbers z_α. They can be projected vertically onto the real axis *Re*. There, marked as white dots, they represent the real part of the complex numbers z_α. The rudimentary curve α can be smoothed by circle segments or Bézier curves.

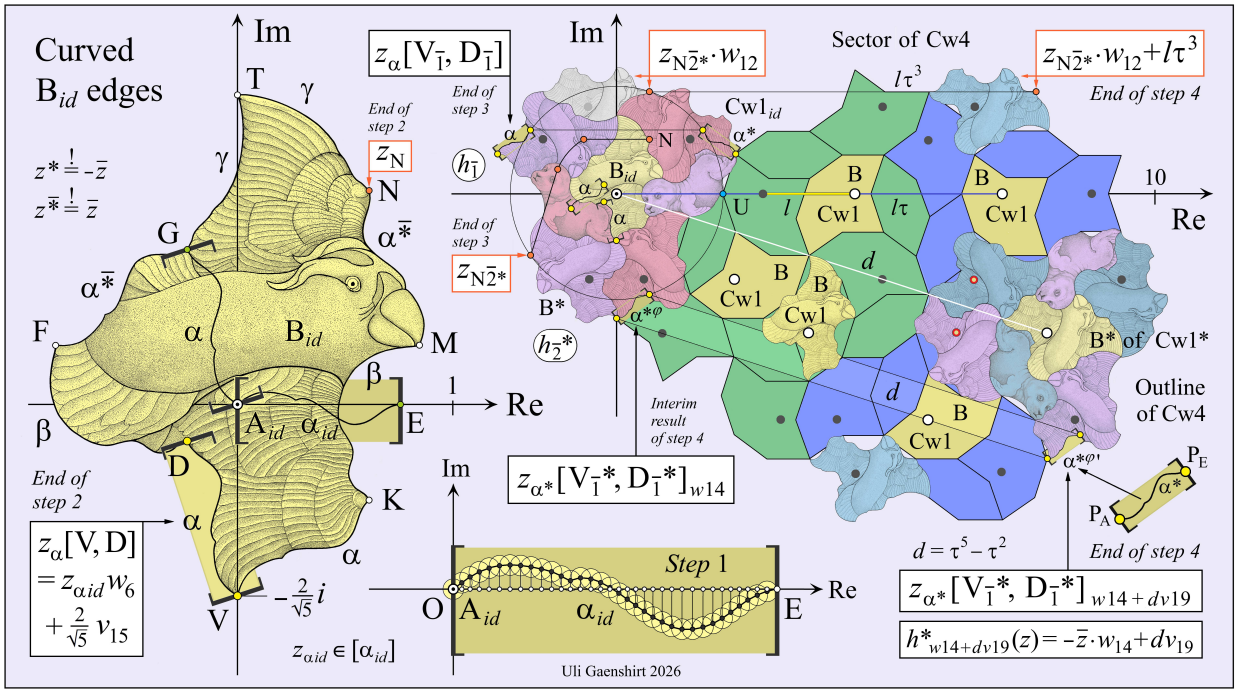

Figure 7: *The four steps of the algorithm shown at the examples of figuratively designed tile edges.*

The yellow bird on the left side of Figure 7 has exactly the same corner point geometry as the yellow *B* tile in Figure 6(a). There, we have already described the edge a_{id} as a set of points in the interval $[A_{id}, E]$ in order to capture the properties of the edge marks. Therefore, the descriptions of the examples in step 2 in Figure 6(a) also apply here, whereby the curved edges are labeled α, β, and γ.

The two examples of step 3 in Figure 6(b) are shown in reduced size at the top middle of Figure 7. Therefore, only the start point *N* and the end point $z_{N\bar{2}*}$ of step 3 are named. The three edges *a* from step 3 in the green intervals in Figure 6(b) are curved edges here. They are simply marked with square brackets, and all three are labeled only with α, so that the indexing can be used exclusively for step 4.

In step 4, the entire cartwheel $Cw1_{id}$ is transformed. We use the equations as described in Figure 5. For the example of $z_{N\bar{2}*}$, we choose the cartwheel *Cw*1 on the far right part of *Re* as the final destination. We rotate $z_{N\bar{2}*}$ by multiplying it with w_{12}. The point $z_{N\bar{2}*} \cdot w_{12}$ belongs to the rotated bird in the background! Then we shift $z_{N\bar{2}*} \cdot w_{12}$ by $l\tau^3$ parallel to the real axis *Re*. The result is the point $z_{N\bar{2}*} \cdot w_{12} + l\tau^3$.

For the example of the curve α with the point set $z_\alpha[V_{\bar{1}}, D_{\bar{1}}]$, we choose the only mirrored cartwheel *Cw*1* in the sector of *Cw*4. So we first have to reflect α about the *Im* axis and obtain α^*. Then we rotate α^* by a multiplication with w_{14} and receive $\alpha^{*\varphi}$ or, more precisely, $z_{\alpha^*}[V_{\bar{1}}^*, D_{\bar{1}}^*]_{w14}$. Finally, we shift with dv_{19} and obtain $\alpha^{*\varphi\prime}$, or, precisely, the set of the complex numbers $z_{\alpha^*}[V_{\bar{1}}^*, D_{\bar{1}}^*]_{w14+dv19}$.

Summary

After an introduction to the geometry of the *HB* tiles and their practical application as puzzle pieces, the main part of the paper outlines an algorithm that makes it possible to digitally generate a quasiperiodic *HB* cartwheel *Cw*4 using three self-designed curved edges, which are transformed by complex equations. In step 1, the edges are converted to point sets. In step 2, the curved edges are transformed onto the edges of the central B_{id} tile. In step 3, this B_{id} tile is transformed onto four satellite tiles of the cartwheel $Cw1_{id}$. In step 4, $Cw1_{id}$ is transformed 34 times by complex equations to form the cartwheel *Cw*4.

Conclusion and Outlook

The graphic artist M.C. Escher (1898–1972) was a globally renowned pioneer in the field of figurative periodic patterns [1]. Today, J. Richter-Gebert's computer program iOrnament enables users to design these periodic patterns with digital support [10]. A similar program for designing quasiperiodic tilings, as proposed in this paper, does not yet exist. The two representative examples, which guided you through the cartwheel algorithm in four steps each, show that this quasiperiodic algorithm can be used as the basis for a planned computer design program.

When designing the tiles, we must be aware that the lines often need to be redrawn, because if one of the edges is improved in relation to a specific part, four or six edges of this type within the two prototype tiles are changed at the same time. That is why designing quasiperiodic tilings promotes synergetic thinking. In this sense, the computer program would have enormous *educational potential* and would be perfectly suited for *interdisciplinary teaching*, without neglecting the fun factor.

The program should be available free of charge. It should be possible to draw the three curved edges (or more) with a pen on an external mobile phone or tablet, which can then be duplicated in the program and displayed on a screen. We will be very grateful for any help with programming, because we have no experience in this area. Support from an educational institution would be great.

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