IMAGINARY

Mathematisches Forschungsinstitut Oberwolfach

Xi'an Jiaotong-Liverpool University
西交利物浦大学

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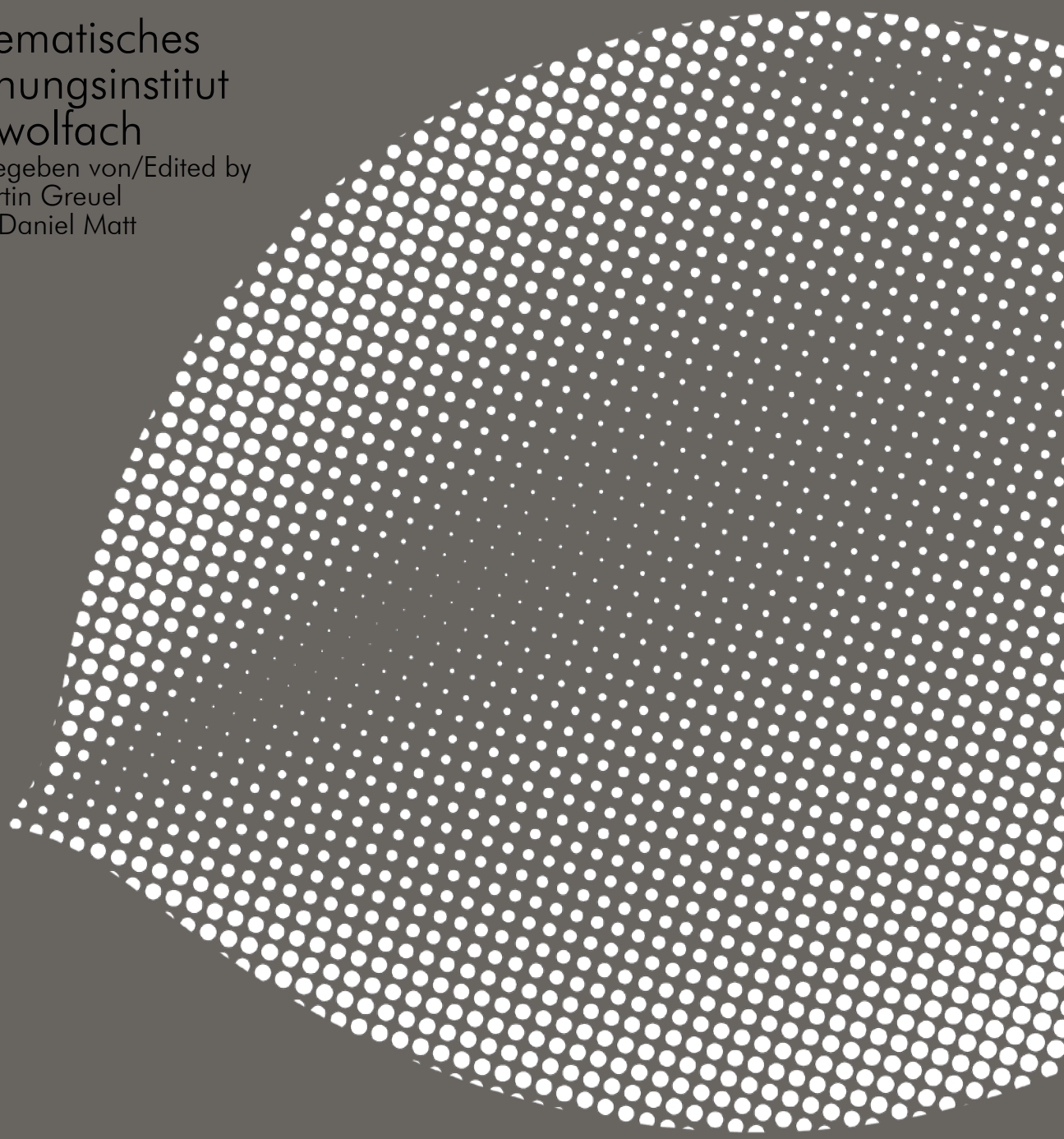

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ISBN 978-3-00-026939-4

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IMAGINARY -

MIT DEN AUGEN DER MATHEMATIK
THROUGH THE EYES OF MATHEMATICS

Katalog zur Wanderausstellung
Travelling Exhibition Catalogue

It is the joy about the form
that characterizes the geometrician.
Alfred Clebsch
(German mathematician, 1833 - 1872)

FOREWORD

Dear readers,

This book invites you to have a look at mathematical creations “through the eyes of mathematics“ in a way you have perhaps never done before. The figures shown are taken from geometry, which is just one field of mathematics, though an important one. Maybe, the word geometry makes most of you think of triangle geometry from school days, but modern geometry is much more. It is of major current importance and, what is more, it is extremely attractive. Today, this attractiveness can be expressed in images generated by mathematical programs by means of computers such as was not possible some years ago.

In a large number of display cabinets we show you amazingly beautiful images, which often obey quite simple formulas. Leg-ends to the images indicate the origin and meaning of the images and suggest some mathematical concepts. The images were com-piled for the travelling exhibition IMAGINARY by internationally renowned mathematicians and graphic artists. The exhibition was prepared for the Year of Mathematics 2008 in Germany by the Mathematisches Forschungsinstitut Oberwolfach.

You certainly wonder what these images of surfaces and their singularities are good for, what they mean, whether they have applications or serve to describe real phenomena. This is true, in-deed, but it is not at all simple to explain. Introductory articles on our website www.imaginary2008.de by various mathematicians try to illustrate some of the real backgrounds or kindly provide a glimpse into the mathematical workshop and finally presents you with a current research problem. Moreover, there are innumerable relationships of algebra and geometry to other sciences, but there are also quite specific applications in industry and economy.
Below, you will find information on the exhibition and our experi-ence with the broad audience. Of particular interest to visitors and media were the programs enabling interactive experience with geometry.

The image displays in this book are accompanied by a short bio-graphy of the artists and an overview of the particular mathemati-cal field.

I wish you an inspiring and informative read and I hope you will enjoy the beauty of mathematics and gain better understanding.

Gert-Martin Greuel
Technische Universität Kaiserslautern und
Direktor des Mathematischen Forschungsinstituts Oberwolfach

Oberwolfach, am 15.11.2008

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IMAGINARY-THROUGH THE EYES OF MATHEMATICS

THE IDEA

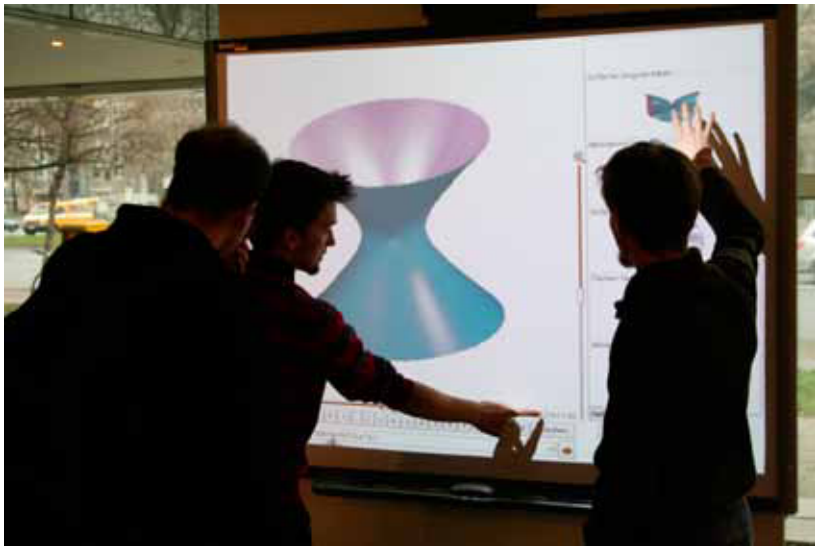
The idea behind the IMAGINARY exhibition is – as the name suggests – to use the visual and aesthetic component of mathematics as an eye-catcher in order to explain the visitors the mathematical backgrounds in an interactive manner. The imaginary and unimaginable of mathematics is illustrated, it turns to images which you can generate yourself.

ALGEBRAIC SURFACES INTERACTIVE

On the one hand, the exhibition consists of a gallery of interesting and beautiful mathematical images taken from algebraic geometry. They are presented at a size of 85 x 85 cm on acryl glass and can be gazed at in a walk-in aluminium cube, the IMAGINARY Cube. Each image is provided with a table which explains mathematical properties and how the image was generated. The important elements of the images, such as singularities, are described. On the other hand, interactive installations at the exhibition invite the visitors to take mathematical artistic action themselves. For this purpose the Surfer program was designed to calculate, indicate and change algebraic surfaces in real time. Visitors can enter and change polynomial equations on a large touchscreen with their fingers, shift parameters, determine the colours of the surfaces and turn the figures as they like.

NTERNATIONAL PARTICIPATION

Herwig Hauser from Vienna presents a film showing mathemati-cal fancy surfaces and artfully presented algebraic surfaces. His images form the core of the IMAGINARY Cube. The film and the images were already very successfully shown at the International Congress of Mathematicians in Madrid in 2006. Richard Palais and Luc Benard (USA, Canada) present elaborate visualizations, e.g. Wada Basins or Triple Periodic Surfaces. The awarded image with a collection of well-known surfaces on glass such as the Boy surface, is also on display. It won the Science Visualization Chal-lenge, 2006, of the Nature Journal. Palais, for years, has been working on the 3D_XLPLORMATH program, a very comprehen-sive interactive program which aims at discovering mathematics from a three-dimensional view. Hermann Karcher from Bonn was among his team members. The program is presented, among oth-ers, with an introduction to be viewed with 3D glasses. Jos Leys, Étienne Ghys and Aurélien Alvarez (Belgium, France) present the first part of their new DVD series DIMENSIONS, which explains the mathematical concepts on a generally understandable level. The first part which clearly presents the stereographic projection and offers a look into the fourth dimension can be seen in public for the first time. In addition, Ghys and Leys put spectacular imag-es on display such as the hecatonicosachoron or the Anosov flow. Ulrich Pinkall and his



group in Berlin participate with the interac-tive JREALITY program, which generates a virtual mathematical world where visitors can move freely, such as climbing on surfaces or falling down. Images of the group contain, among others, an artful torus or a tetranoid. From Berlin comes the multiply award-ed film MESH on discrete mathematics made by Konrad Polthier and Beau Janzen. Jürgen Richter-Gebert from Munich prepared a series of interactive applications for IMAGINARY based on the program CINDERELLA. They communicate various mathemati-cal topics as simulation, chaos or symmetries in a playful way. Martin von Gagern's program MORENAMENTS allows to paint symmetrical patterns in one of the 17 space groups in the Euclidian plane. The SURFER program based on the Surf program by Stephan Endrass was specifically developed for the IMAGINARY exhibition by Henning Meyer from Kaiserslautern and Christian Stussak from Halle assisted by Oliver Labs from Saarbrücken.

THE EXHIBITION – SURFACES AS SCULPTURES
The companies Voxeljet Technology at Augsburg, and Alphaform at Feldkirchen are engaged, among others, in generative manufacturing of 3D models by selective gluing plastic powder (PMMA) or by means of stereolithography. Both companies have accepted the challenge to print in 3D a selection of algebraic surfaces of the exhibition and present them as sculptures. The challenge was to find appropriate model data records. Therefore, the Institute FORWISS of Passau records. University implemented various techniques for the exhibition to translate algebraic surfaces into printable data. Ten sculptures at a diameter of about 25 cm are exclusively displayed at the exhibition. They were produced for the exhibition by the two companies free of charge.
SUSTAINABILITY In order that the exhibition and also the contents continue to be used sustainably a didactic package together with manuals and software is established and is available for visitors and also schools. All are invited to also use the Surfer program from home and generate interesting surfaces. A mathematics-art-competition motivates the visitors to take action themselves also after the exhibition. The exhibition will continue to be open in 2009. Parts of the exhibition are to be installed in German museums permanently, among others, in M2 – Mathematik und Mineralien Erleb-niswelt Oberwolfach / Oberwolfach mathematics and minerals world of experience – which will be

opened in 2010 .SUPPORT
The IMAGINARY travelling exhibition is supported by the Federal Ministry of Education and Research. A number of private persons, firms, universities and institutions have campaigned for the exhibi-tion to be implemented in the Year of Mathematics. Thank you very much to all who supported the exhibition!EXHIBITION VENUES IN 2008In 2008 IMAGINARY was shown in 13 towns and in a large number of actions and special exhibitions all over Germany. At almost all places there

was a cooperation with local universities in the area of science and didactic support.

Munich, TU Garching	10.12.2007 - 18.01.2008
Berlin, Urania	19.02.2008 - 06.03.2008
Kaiserslautern, ITWM	10.03.2008 - 11.04.2008
Stuttgart, Ideenpark	17.05.2008 - 25.05.2008
Potsdam, Train station	29.05.2008 - 25.06.2008
Leipzig, Science Summer	28.06.2008 - 04.07.2008
Rust/Freiburg, Europapark	08.07.2008 - 30.07.2008
Kassel, Sparkasse	05.08.2008 - 29.08.2008
Cologne, Uni-Mensa	08.09.2008 - 26.09.2008
Munich, LMU	24.09.2008 - 21.10.2008
Constance, BildungsTURM	01.10.2008 - 19.10.2008
Saarbrücken, K4 gallery	24.10.2008 - 16.11.2008
Passau, University	05.12.2008 - 19.12.2008
Special exhibitions	
Munich, Dies Academicus	05.12.2007 - 06.12.2007
Berlin, Opening gala	23.01.2008
Berlin, TU	28.01.2008 - 05.02.2008
Berlin, Girls' Day	24.4.2008
Science ship	07.05.2008 - 04.09.2008
Bremen, Kino46	24.05.2008 - 26.05. 2008
Koblenz, Science Day	09.06.2008
Halle, Science Night	04.07.2008
Mannheim, Computermarkt	19.07.2008
Salem (USA), Robert Bosch	27.07.2008
Jena, Junior Math Congress	04.08.2008 - 08.08.2008
Science Days LMU	18.10.2008 - 21.10.2008
Cologne, Closing Ceremony	11.12.2008
Exhibition Second Live	14.12.2008 - 01.02.2009

ACTIONS AND PROJECTS

MATHS CASE: Together with the MNU two posters and three info leaflets were produced for the Maths case (circulation 2500 pieces) containing ideas of how to apply the IMAGINARY pro-grams at school.

DFG-DVD: Together with pictures and background information IMAGINARY is included in the DVD of the German Research Institute which was compiled for the Year of Mathematics.

POSTER SET: 12 motifs of the exhibition (algebraic surfaces) with information text are offered as IMAGINARY poster set for a service charge. More than 300 poster sets have already been ordered.

COMPETITIONS: Together with media partners (Die Zeit, Spektrum der Wissenschaft) mathematics-art-competitions took place with Surfer program from February to December 2008. More than 8000 images with their formulas were produced for the competitions altogether.

IMAGINARY AT SCHOOL: Schools are involved in using the pro-grams of the exhibition at school. For this purpose a website was compiled where ideas of how to use them are collected.

EUROPEAN MATHEMATICAL SOCIETY: A comprehensive report on IMAGINARY was established, which was presented at a conference on the subject "Raising public awareness in Mathematics" at Utrecht, Netherlands, in July 2008.

MINI SYMPOSIUM AND TEACHER TRAINING: A mini symposium was organized at the DMV meeting at Erlangen on the subject IMAGINARY (Techniques from visualization in geometry) on 16 and 17 September 2008. A teacher training on "theory and visualiza-tion of algebraic curves and surfaces" took place at Oberwolfach from 02.11.-08.11.2008.

www.dimensions-math.org
www.mesh-film.de

EXPERIENCE IN THE YEAR OF MATHEMATICS 2008

The year of mathematics 2008 was successful beyond all expectations. Mathematics has caused enthusiasm, in numerous events, at lectures, in films and plays. One of the highlights of the year of mathematics was the travelling exhibition IMAGINARY which aroused very much audience and media interest. Below we present you some experience which we made in the course of our work.

KEY FINDINGS:

MATHEMATICS IS ATTRACTIVEAudience feedback to the year of mathematics in Germany has clearly shown: Mathematics is an attractive subject - despite its reputation of being dry, difficult and boring. People are very interested in understanding it, in thinking mathematically and solving mathematical problems. There is prejudice that maths is not attractive to a broad public – it is rather the opposite when explained in a clear and understandable manner.

THE BEAUTY OF THEORY NOT EVERYTHING HAS TO BE APPLIED
Our experience with visitors of IMAGINARY has shown that pure mathematics - presented in the right manner - is very appealing. There is no need to link it to applications. The question "Where can we apply this?" was raised mainly by journalists.

"WE WANT TO DO IT OURSELVES" INTERACTIVITY AND PARTICIPATIONA
key element of our success is to make people feel that they can actually do (complex) mathematics themselves, create their own formulas or their own images, which are equally beautiful as the ones created by mathematicians. The online competitions were extremely well accepted. People like to create, take part, interact, participate and especially showcase their own work (and get feedback).

OPEN AND FREE APPROACH TO SOFTWARE AND CONTENT
All software used at the exhibition is available for free and home use. Many of the images presented at the exhibition are provided in high-resolution for visitors to make their own prints. This flexible approach towards the content of the exhibition brought many users to the website and resulted in very positive feedback from the public. The programs have been downloaded more than 40 000times. We got many requests from design offices, companies, schools, etc. to use IMAGINARY software in a variety of fields.

RAISING AWARENESS ON MANY FRONTS
To do maths exhibitions involves much more than the exhibition itself: website, media work, free software, creative competitions, individual requests, general texts on mathematics for media, didactic material for schools, etc. The joint impact of all these ac-tions leads to a successful project.

NETWORKING AND PARTNERS – FUNDING AND FAME
To cope with the "many fronts" explained in the previous point, it is necessary to involve local universities/research institutions and other organizations (for media work, students support, etc.), big national media, politics and companies. Difficulties that arise may be: partners' own interests, dependencies of partners, desired individual media presence of participants, issues of money and competition in acquiring funds. Since many of the contributions are done voluntarily by the science community, it is very important to keep a good balance of "who is presented in the media and in which way". The commercial aspect of these activities (funding, sponsoring, selling merchandising products, entry fees, etc.) can create conflicts.

INDIVIDUAL GUIDANCE – THE HUMAN SIDE OF SCIENCE COMMUNICATION

Our experience in communicating mathematics on site showed that individual guidance is fundamental to dissolve prejudices towards the subject. The tutors take their time to individually explain details adapting to the level of knowledge and interest of the visitor. It is very important that these tutors are well trained and transmit their own passion for the subject. What is of special importance is the language used – for the introduction specific mathematical terms should be avoided (for example "swimming belt or doughnut" instead of "torus"). It is important that the in-teraction is simple and intuitive. For those who want to go deeper the tutors should know some of the mathematical background or the use of the theory in other disciplines in order to answer ques-tions. People are eager to learn.

FEEDBACK TO SCIENCE

Science communication also reflects the current state of scientific results. Some of our programs and techniques resulted in new algorithms and more insights in visualization, 3D printing tech-niques, etc. A mini-symposium on these results was organized at the conference of the German Mathematical Society DMV in fall 2008.RESULTS- Exhibitions held in Berlin, Munich, Kaiserslautern, Stuttgart, Potsdam, Leipzig, Europapark Rust, Kassel, Cologne, Constance, Saarbrücken and Passau- 340 school classes, 120 000 visitors of the exhibitions, another 100 000 visitors of the special exhibitions- Website: on average 400 unique users daily (with occasional peaks, for example 9000 unique users on one day) and more than 5 000 000 hits- 40 000 downloads of our SURFER software, 25 000 downloads of articles on the mathematical background written for a broad public- Many small and big articles in local, national and international newspapers, reports on radio and TV

SOFTWARE AND IMAGE COMPETITIONS

The exhibition IMAGINARY offers free programs for mathematical visualization:

3D_XplorMath is one of the most comprehensive programs to visualize mathematical objects. Developed by the 3D_XplorMath Consortium.

¡Reality, at first sight, looks like a computer game. You run through a virtual world to experience mathematical objects. Developed by TU Berlin.

Cinderella: Create easily and quickly geometrical constructions or a virtual physics simulation. Developed by the Cinderella Team.

morenaments: Paint intuitively symmetrical patterns in one of the 17 space groups in the Euclidian plane or even hyperbolic orna-ments. Developed by Martin von Gagern.

Surfer is made to easily visualize real algebraic surfaces. Devel-oped by TU Kaiserslautern for the exhibition IMAGINARY.

The interactive programs allow playful contact with geometry, i.e. they encourage the users to create formulas themselves or to change already existing ones in a creative manner. It is surprising to see how amusing it is and how you begin after some time and pondering to understand the role of the underlying mathematics.

Together with Zeit Online and Spektrum der Wissenschaft mathematics-art-competitions were organized the aim of which was to award and collect interesting and beautiful images being generated with Surfer program. More than 8000 transmittals with amazingly beautiful images are evidence of the success of these competitions. On the next page you will find a small selection of transmittals.

All programs are available for free on the IMAGINARY website.

http://surfer.imaginary2008.de
www.zeit.de/matheskulptur
www.spektrum.de/mathekunst
www.imaginary2008.de/galerie

黑森林中的数学胜地

简记奥博沃尔法赫数学研究所之传统与文化

奥博沃尔法赫数学研究所全景

我曾有幸在一些著名的数学研究所或中心短期学习。它们各具特色，总是我心中美好的记忆。德国波恩大学豪斯朵夫（Hausdorff）数学所隐于泊波尔斯多夫宫（Poppelsdorfer Schloss）前长长的草地边上一典雅的小楼，犹记行走在楼梯的木板上会咯吱作响。我造访过英国剑桥牛顿（Isaacewton）数学所，它和图书馆等许多崭新的小楼组成的建筑群（剑桥数学中心）座落在“乡村”中， 袭人的漂亮。我也到过意大利Cetraro掩于海滨山腰的国际数学暑期中心，那里盛开着烂漫的红花，整山遍满着绿色的葡萄架；特别令人难忘的是中、晚餐要乘电梯下到深深的悬崖脚一临海小餐馆。而我最近前往学习了一周的德国黑森林中的奥博沃尔法赫（Oberwolfach）数学所无疑是其中极具魅力的一个。这里不但是一个清秀可人的胜地，而且也许是所有青年数学家一生中都值得去拜访一次的“麦加”圣地。

我第一次听说奥博沃尔法赫数学所是两年前有同事邀我一起申请去那里学习，但我因故未去；后来有老师也建议我们应该去看看，还特别提到那里的图书馆。我且简记我所粗略了解的奥博沃尔法赫数学所，与广大数学爱好者一起来感受这个数学所的传统与文化。

今昔

奥博沃尔法赫数学研究所位于德国西南部巴登-符腾堡州（Baden-Württemberg）黑森林中小山村 Oberwolfach-Walke上的一山腰。黑森林是德国最大的森林山脉（长约200千米，宽约60千米），山上森林密布，远望黑压压一片，因而得名。数学所德语名称为Mathematisches Forschungsinstitut Oberwolfach，缩写为MFO。“Ober”在德语中有“上方、高处”的意思，所以该数学所也被人称为上沃尔法赫数学研究所。习惯上人们常以地名“奥博沃尔法赫”称呼它。

二战中期，德国为加强军事力量而创立了几个国家研究所。时任弗莱堡（Freiburg）大学校长、德国数学会主席的数学家威廉•聚斯（Wilhelm Süss）利用此机会于1944年创立了奥博沃尔法赫数学所。主要为了躲避可能的空袭，聚斯选中黑森林中一个为打猎者服务而建的小旅馆Lorenzenhof作为数学所。历经发展，数学所现在由三个主要建筑组成，一为宿舍楼（含咖啡厅），一为图书馆和会议厅，另一为较长期的研究人员的宿舍。

现在看来，选择这样一个至今仍远离尘嚣的地方反而成为奥博沃尔法赫的一大特色。短期中和志同道合的同事独享这里的孤独，该是很惬意的事情。而且数学所事实上并没有为纳粹服务，却通过数学家的合作，在战后为改善德国与其他国家破缺的关系提供了帮助。关于数学所的详细信息和历史，可以参考其主页1以及文[1, 2]等资料。

Oberwolfach-Walke小村

学术

奥博沃尔法赫数学所与其它很多数学所，如德国波恩的马克斯•普朗克数学研究所（Max-Planck-Institut für Mathematik）不同的是它没有永久固定的研究人员（除主任外）。奥博沃尔法赫只有少数短期研究人员及部分工作人员，如“Research in Pairs”（RiP）项目资助各学科二到四名来自不同地方的学者结伴在奥博沃尔法赫进行为期两周至三个月的合作研究。奥博沃尔法赫主要以会议的方式逐周接待访问学者。通常如果研究人员固定，则研究兴趣也往往局限于某些方向。与奥博沃尔法赫数学所研究人员的流动相应的是它的研究兴趣的多样性：不但涉及数学各学科，也旁及和数学有交叉的诸如物理、生物、医学以及天文等领域。

从1949年到1953年奥博沃尔法赫每年仅组织三到五次会议。后来随着条件的改善，特别是自1969年新建了客房后，会议数逐渐增多。现在除圣诞和新年假期外，每年52周中有50周在组织会议。迄今为止，它已经组织了好几千次会议。奥博沃尔法赫的网页上列有它历年所组织的会议的详细信息。基于一些会议，一系列有影响的论文集等著作也随之出版。

奥博沃尔法赫的住宿每周可以容纳55-60名访问者。其中除了最多有10名是参与RiP项目的研究员外，其它人（约60%是来自德国以外的地区）都是受邀前来参加每周的会议的。

奥博沃尔法赫因人数的限制，因此每周或组织一个共约45至48名专家参加的研讨会（Workshop），或三个平行的16人左右的迷你型研讨会（Mini-workshop），或两个约24人的讨论班（Seminar）。此外，每年春、秋两季（一般分别在4月初和10月初）各举行一次45人参与的“学习群”（德语“Arbeitsgemeinschaft”，英文“Study Group”）的会议。

奥博沃尔法赫还有一个针对巴登-符腾堡州的中小学教师和图书馆员的培训周。同时，它还为准备参加奥林匹克数学竞赛的学生进行为期一周的最后训练。

一般各领域的数学家可以向奥博沃尔法赫提前申请组织会议。对各会议感兴趣的数学家，除学习群的活动是向组织者申请外，都需直接向数学所主任申请。

研讨会给专家们提供机会报告自己研究的最新结果和新方法，并开启将来的研究计划。迷你型研讨会的主要目的是刺激新的研究：如联合研究力量对某问题进行攻关，或共同引领特别有趣的发展。为这样的目的，小型团队往往较合适，而年轻人作为新鲜血液也被强调给予机会来参与甚至组织这样的活动。

Wolfach小镇

沃尔法赫的街道

我所参加的是一个讨论班，类似于一个短课或暑期学校。两位组织者系统地介绍马氏过程遍历论以及他们在该领域取得的最新成果。他们同时还邀请了3位相关的专家来做报告。我们学习时间安排较紧，除周三下午外，每天安排有共5个多小时的报告，且下午开课前为自由讨论、答疑时间。

学习群的传统自上世纪50年代就开始了。其想法类似于布尔巴基讨论班（Séminaire Bourbaki）,选定某论题，以讲课的方式来学习新近由其它数学家在该领域取得的结果。例如，2009年度春季学习群的主题是“最优运输”2。一般参与者向组织者申请加入，而第n+1次的组织者及主题则由第n届活动的人员投票推荐。

奥博沃尔法赫将它比作流水式热水器（a continuous-⊠ow water heater）。它将数学家邀请来，提供理想的条件，进行密集的研究活动以影响、刺激各研究领域的发展。对于年轻的博士生，这里是很好的新兵训练营。

旅途

一般每周日下午各地与会人员乘穿行在山谷中的火车，先到达小镇Hausach或再前行抵达沃尔法赫（Wolfach）。我是从比勒费尔德（Bielefeld）一路南行经过6个多小时的火车转乘先到沃尔法赫。沃尔法赫实在是一个小得可爱的小镇，有着一条不长且整洁古朴的步行街；与此相垂的是一条山谷小河，民舍傍河而筑。这里通行的列车车厢不长，铁轨也是来往共用。未到奥博沃尔法赫，即可以预先感受到数学所的宁静。沃尔法赫距离数学所约5公里。到访者也可以乘一小时一趟的公交车到Walke站再步行上山或坐出租车。与会的人员一般在周六上午离开。我参加的讨论班的与会人员离开时，奥博沃尔法赫请出租车按各人的离开时间分批免费接送到火车站。据最新的通讯消息，从2011年起奥博沃尔法赫对所有来访者提供免费的出租车接送服务。

住宿楼

资助

奥博沃尔法赫提供免费食宿。我这次行程，因西门子基金对2008年夏到2013年夏期间奥博沃尔法赫讨论班的支持，还可以申请到200欧元以内的旅费资助。奥博沃尔法赫作为一个非盈利机构，也接受各种赞助。图书馆里挂有小纪念牌说明哪段时期何公司或基金提供了资助。有的国家的相关机构也有专项经费设在奥博沃尔法赫以资助其国家的数学家到这里进行研究和学习，如美国国家自然科学基金（面向青年数学家）、日本数学会（青年数学家优先）。我看到美日的数学家能有这样的资助的说明时，自然想起有关陈省身先生的一个故事。陈先生把获得“邵逸夫科学奖”的100万美元奖金捐给了一些国家的数学所，附信说：“希望将来在中国数学家到贵所时能给以更多的照顾”（见演讲[3]）。

食宿

最早期的到访者的食宿相当简朴，须自带伙食，并且要亲自收集柴火。现在的奥博沃尔法赫仍有着家的气氛。住宿宽敞、简洁而舒适；红色的地毯垫出满屋的温暖。住宿楼层里还有巨型国际象棋摆在地面。一日三餐都在底层的咖啡厅。除早餐较固定外，午餐、晚餐的食物几乎每天每次都不雷同，可以尝到许多别具特色的食品。很感谢数学所为数不多的工作人员。

图书馆

图书馆一共两层，上层主要是报告厅，下层主要为图书收藏与研究、文娱场所。图书馆里面有黑板、书桌、复印机和计算机等设备；也有运动、娱乐场所和器材，如乒乓球室、台球桌以及置有钢琴、提琴等乐器的音乐室。

图书馆主要收藏数学相关的文献，其量之丰，去访问过的人无不称赞。据奥博沃尔法赫网页的介绍，按美国数学会的评价，它的图书馆排名第十。它目前收藏有45,000多本书，27,000多卷期刊，且订有500多种纸版期刊以及3,000种电子期刊。与其它图书馆开始更多地转向电子期刊订阅相比，奥博沃尔法赫订的纸版期刊是很多的。图书馆这样良好的基础设施，也极大地吸引着数学家们前往合作研究：不但无需携带所需资料前往，或许还可以找到以前较难寻获的文献。

图书馆的另一特色收藏是许多精心挑选的数学软件。在现代数学研究和教育中，数学软件所起的作用已经不亚于各种文献。该收藏的目的是建立一个永久的数学软件基地，使得软件的查找、使用和评价等更加有效。读者在其主页3可以浏览各分类介绍的软件。

图书馆的咖啡机免费使用。我在奥博沃尔法赫的留言薄上看到，此前有不少人留言建议添置咖啡机；正佐证了厄多斯（Paul Erdös）的名言“数学家是将咖啡转换成定理的机器”。其它饮料，如矿泉水、啤酒等只收很少的费用。

住房

交流

学科交叉与合作研究在现代数学的研究中越来越突出。虽然互联网的发展使得交流越来越容易，但数学家们还是很乐于面对面的交流，因为这样的交流所得有时可能比书斋中独自冥思苦想更有效。此外，数学家们从初次见面到相互认识、成为朋友也往往为将来的合作奠定基础。

我参加的那次会议有来自德、英、美、韩国以及俄罗斯、瑞典等国的；来自中国的一共有4人（都在欧洲学习）。宿舍没有电视、 电话，也没有网线，而且咖啡厅的无线信号在房间中也很弱。其假设是与会者都在咖啡厅互相讨论，而不是留在各自房间里。咖啡厅晚上放有很多水果、面包等，都免费自用。下午正式会议之前为蛋糕、咖啡时间（这也是很多数学所的习惯，如Hausdorff数学所下午有休息时间，提供免费点心）。

一起就餐是很好的交流机会。很多西方大学的研究组都有一个很好的传统——相约一起前往午餐。奥博沃尔法赫就餐时的座位分配很能促进交流。除早餐按个人习惯或早或晚外，中餐、晚餐时间统一且座位是随机安排的。就餐时须找到带有自己名字标签内盛餐巾的小布袋入座，其目的是尽量使不同的人有机会同桌共餐、互相了解。

远足

会议从周一到周五上下午一般都安排有报告，但周三下午是传统的远足活动。愿意一起远足的人，中饭后不久就要出发，凭借着一张简易地图，爬山、步行约两个多小时到达一个小餐馆。休息约一小时后，再步行回去。返回数学所时，灯火初上，已是晚餐时间。长途远足有益身心，还可以在一种更加开放的环境里互相交流。

年轻人

奥博沃尔法赫特别鼓励年轻人的申请。比如讨论班的主要参与者是年轻的博士生和博士后，并如前述还有旅费资助。目前奥博沃尔法赫各会议有特别的经费资助博士研究生。奥博沃尔法赫也提供短期的博士后职位给取得博士学位5年之内的博士，如“Oberwolfach Leibniz Fellows”。此外，针对年轻人还设有数学奖，著名的有奥博沃尔法赫奖，候选人的博士学位要求为10年之内取得的。因证明了朗兰兹纲领自守形式中的基本引理而获得2010年度菲尔兹奖的越南（/法国/美国）数学家吴宝珠（Ngo Bao Chau）即获得过2007年度的该奖。

开放的环境

奥博沃尔法赫所有住宿房间没有钥匙，离开房间不能也不用锁门。除了周六午后、周日午前这个数学所基本上没人的时间，图书馆、宿舍大门也都不锁。啤酒等部分收费饮料是各人自取，只在一纸上自行计数，离开前将钱放到一盒中结清即可。这样一种开放的环境，在这里是如此自然，不但是人与人之间相互的信任和尊重，也是促进彼此交流的催化剂。其实，与此类似，很多大学，不仅仅是数学系，许多人的钥匙是主钥匙，能开很多办公室的门。我们读下哈尔莫斯（Halmos）在他的数学传记《我想作数学家》中一段描述和评论（参考[4]，原著第130页，译著第168-169页）就足以理解这样的传统的高贵了：

“系秘书给了我一把钥匙，一把主钥匙！它可打开Eckhart大楼中所有的数学办公室。这是一个古老而高贵的传统，增进了系里同僚间的气氛。我们都是学者，家庭中的一员，我们尊重他人的隐私，但正如一个家庭，我们彼此不设防。每个人的图书都对别人开放，如果我从图书馆借来一本杂志，你随时都可以来翻阅你要的文献。其它大学的访问学者可以在某人的办公室里躲上几个小时。主钥匙常被使用，但从未被滥用，这是一个很好的管理方法。”

数学家图库

奥博沃尔法赫维护着一个著名的图库4，主要是著名数学家及数学活动的照片。这个图库基于Konrad Jacobs教授捐的他所收藏的大量世界各地数学家的照片，也包含了George M. Bergman等教授拍摄的数学家图片。奥博沃尔法赫现在不但公开征集数学家或数学活动的图片，而且自1998年开始就利用数学家来访的机会为他们摄影存档：包括全体参会人员的集体照和组织者的个人照片。譬如，在2010年数学家大会获得菲尔兹奖的四位数学家林登施特劳斯（Elon Lindenstrauss）、吴宝珠（Ngo Bao Chau）、斯米尔诺夫（Stanislav Smirnov）、维拉尼（Cédric Villani）都曾在这里留下身影。

手写的历史

手写的东西总是比计算机敲出来的亲切。奥博沃尔法赫有着一些与手写有关的有趣的传统：

1. 著作签名：参会的数学家被邀请在其被图书馆所收藏的著作上签名；

2. 《摘要》与《问题》：咖啡厅里有两本厚厚的《奥博沃尔法赫摘要》（Oberwolfach Abstracts）、《奥博沃尔法赫问题》（Oberwolfach Problems）。前者记录每个会议的组织者手写的摘要及参会人员的签名；而后者包含有来来往往的数学家留下自己的问题，对前人的评论、问题做出的自己的注释和解答。即使在因特网已经使得这样的交流极其方便的情况下5，这样一本问题集的存在至今还是很有意思的。

3. 建议簿：咖啡厅入口旁的书桌上有一本供来访学者留言的小册子。读者可以留下反馈、评论以及建议等。我看到的那本大约是从1986年开始记录的，其中不乏对数学所的赞扬，对工作人员的感谢，也有很多有益的建议以及有意思的对话。意见簿本身也构成奥博沃尔法赫历史的一部分。

Boy曲面

图书馆和宿舍之间草地上的有一由2厘米宽的钢片编成的Boy曲面模型。这个曲面很受欢迎，如现在用谷歌搜索关键词“Boy Surface”可以得到约28,400,000个结果。Boy曲面由德国数学家Werner Boy在他1901年的论文中作为实投影平面在三维空间中的浸入而首次构出。这原是Boy的导师希尔伯特留给他的作业（证明相反的结论）。

Boy并不知道此曲面的视觉呈现效果。直到1978年Bernard Morin才借助于计算机的辅助得以找到该曲面的第一个参数化表示。文[5]中介绍了许多曲面的参数表示及图形，其中包括Boy曲面，以及同为非定向曲面且同样有趣的克莱因瓶、莫比乌斯带的参数表示和图形。奥博沃尔法赫的这个模型建于1991年，具有三重旋转对称性且使得曲面的Willmore能量最小6。

漂亮的科普

奥博沃尔法赫数学所不但为数学家们的合作研究提供得天独厚的环境，也为普通大众走近数学而不遗余力。

从2000年开始德国每年确定一个主题，举办科学年活动以唤起公众，特别是年轻一代对科学的兴趣。2008年被定为德国数学年。奥博沃尔法赫为此策划了名为“IMAGINARY”7的流动数学与艺术展8。该展览以图片、互动和立体电影等形象直观的方式使公众能“看”到数学，并有许多数学家撰写的普及文章介绍数学背景（如[6]）。展览不但在德国各地展出过，还在加州伯克利、巴黎、维也纳和剑桥等地展出过。

展览中特别有趣的是一系列由各代数方程对应的各种形状的几何曲面，如海马、苹果、红心等形状优美的图形。同时它还提供得到这些曲面的软件Surfer9让参观者以“玩”的方式进行“几何”体验。例如，方程

$1.2(x^2+z^2) = 5y^3(0.25\,y^2)^3$

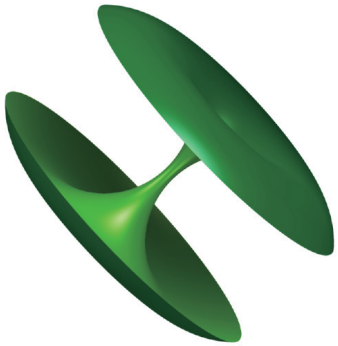
的解(x, y, z)组成柠檬状的曲面。如读者希望看到更多图片，可以到其主页欣赏或参考Surfer的图库。他们也将部分曲面制成漂亮的海报出售。如我所在大学数学系的走廊即张贴了许多。

在Surfer中你不但可以对曲面进行着色、移动以观察曲面的各部分，还可以交互式地自行探索各种代数方程所对应的曲面以及在系数变动情形下曲面的变化。左图为Surfer图库的中“环面”（亦可参考[6]）

$x^2+y^2+z^2+a^2b^2=4b^2(x^2+y^2)$

通过取不同的参数a, b得到。

由左图可以看出，某些方程有奇异点（对应曲面的“尖点”）。按Surfer软件的介绍，d阶代数方程可能产生的奇异点的最大数目μ(d)当d 7时至今未知。许多带奇异点的图片也可在Surfer软件的图库找到10。



MiMa博物馆

奥博沃尔法赫联合其它部门于2010年1月底在数学所附近新成立了一个MiMa博物馆，即矿物与数学博物馆（Minerals and Mathematics）的简称。因为时间关系，我没能去参观。其主页11介绍了部分内容。MiMa主要展出奥博沃尔法赫的两大特点：独一无二的黑森林矿藏和数学所指导下的数学知识。博物馆也特别重视矿物的晶体结构与数学的联系（其中蕴含如准晶、彭罗斯拼图等热点研究问题）。同时，MiMa也是上述“IMAGINARY”展览的永久展览地。

从展览到博物馆，一个世界著名的数学所，将科普做到如此的精致和用心，是很值得我们尊敬和学习的。

影响

奥博沃尔法赫数学所，这个成长在小山村里的数学所有如广袤黑森林里璀璨的明珠，以其充满魅力的传统和文化，吸引数学家们纷至沓来；亦如星星之火，点燃了数学家们建立类似的数学所的激情。其中尤其著名的有法国马赛Luminy的国际数学会议中心12，加拿大的阿尔伯塔省的班夫（Banff）国际数学创新与发现研究站13等。马赛的数学中心早在1954年就被提议设立，但直到1982年才召开第一次数学会议（见[7]）；而班夫研究站则是在2003年正式对世界科学共同体开放的（见[8]）。这些数学中心在介绍其创始历史或缘由时无不提及受到奥博沃尔法赫数学所的启发。

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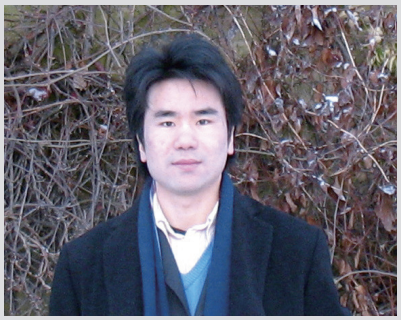

作者介绍：

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ALGEBRAIC SCULPTURES

In order to illustrate an algebraic surface in a 3D printed form we first need a dense triangle mesh, which describes the surface of the subsequent sculpture. In generating these triangles it is essential to preserve particularly beautiful vertices and sharp edges. In addition, having regard to the properties of materials and machine resolution, infinitely thin surfaces must be imitated in the model to be printed by walls which are sufficiently strong, but as detail preserving as possible.

3D-printers pursue the idea to assemble three-dimensional models by layers located one on top of the other and connected with one another. Starting from the model's CAD data the equipment of Voxeljet Technology GmbH, Augsburg, can build, by means of finest adhesive drops of binder sprayed into a bed of plastic powder (plexiglass pearls), a 45 cm high component of more than 3 000 individual layers within 24 hours.

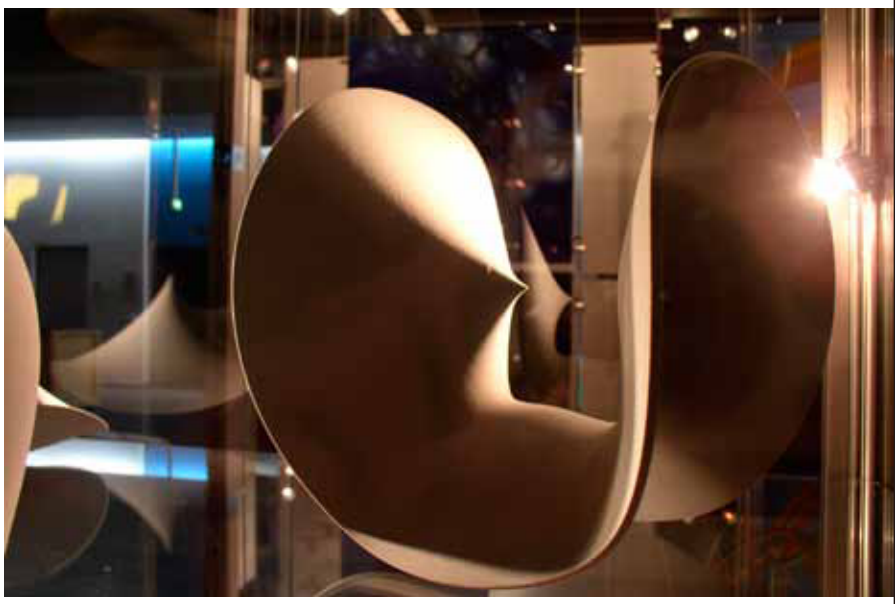
Finally, the bridging between algebraic surfaces and modern digital mould construction ends up with the Finishing of the extremely delicate geometries which are now getting their colouring in the hands of the model maker.

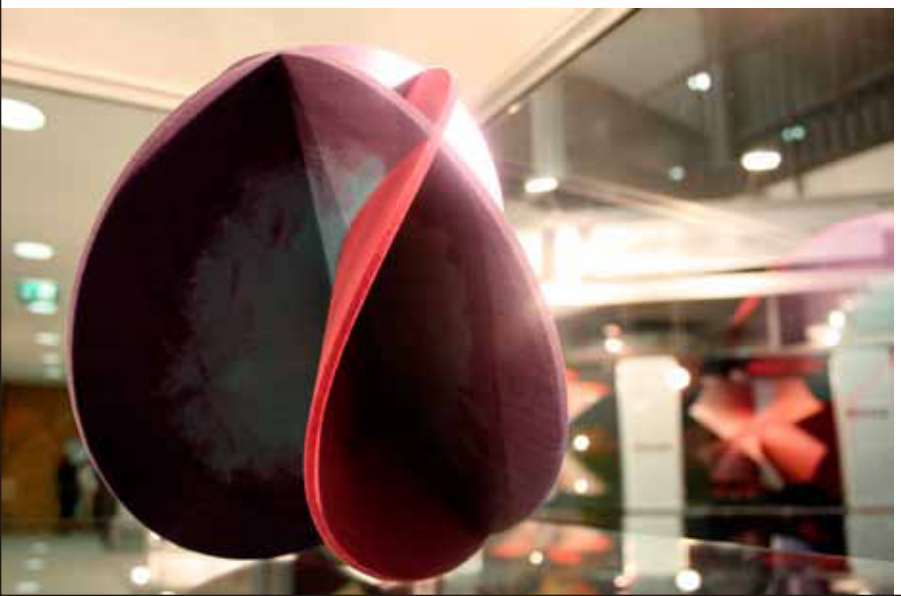
代数雕像

为了在三维格式下打印、展示一个代数曲面，我们首先需要一个稠密的三角形网格，描绘出随后产生雕像的那个曲面。 在生成这些三角形时要特别注意保留优美的顶点和清晰的边。另外，考虑到材料的属性和机器的分辨率，很薄的曲面必须模仿成一堵足够结实的墙壁，同时保留尽可能多的曲面属性。

三维打印机通过层和层之间的高低搭配及连接来装配三维模型。从模型的CAD数据开始，Voxeljet Technology GmbH, Augsburg设备通过装订器上精细的几滴粘着剂涂抹在塑料粉末上（有机玻璃珠），在24小时内制作出3000多独立层的45厘米高的组件。

最后，代数曲面和现代数码制模的桥接终结于极度精美的几何，其着色现在由制模工手工完成。





HERWIG HAUSER

Herwig Hauser is professor for algebraic geometry and singularity theory at the Faculty of Mathematics at Vienna University. He has been concentrating for years on visualizations, films and books for a broad audience. Together with his group he organized exhibitions and lectures on algebraic visualization, his film Zero Set was shown, among others, at the International Congress of Mathematicians, Madrid, 2006.

HERWIG HAUSER

Herwig Hauser是维也纳大学数学学院研究代数几何学和奇点理论的教授。他多年来着眼于面向大众的可视化资料、电影和书籍的制作。他和他的团队一起组织了可视化代数的展览和演讲，在2006年Madrid国际数学家大会上播放了他的电影《零位调整》。

OLIVER LABS

Oliver Labs is a mathematician at Saarland University; his research is focused on algorithmic algebraic geometry and singularity theory as well as their applications. Oliver Labs, for years, has been dealing, in particular, with the construction and visualization of singular algebraic surfaces, among others, also films and pictures for a broad audience. He has already developed several computer programs on this subject, together with co-authors, e.g. Surfex and Surfer.

OLIVER LABS

Oliver Labs是Saarland大学的一位数学家；他的研究方向是算法代数几何与奇点理论及其应用。多年来，Oliver Labs致力于面向大众的，奇异的代数曲面的构建和其可视化，以及电影和图片的制作等。他和他的合作者发展了数种相关的计算机程序比如：Surfex，Surfer等。

ALGEBRAIC GEOMETRY

Algebraic geometry deals with the interplay between Algebra, i.e. the formulas, and Geometry, i.e. the forms. The exhibition shows a selection of algebraic surfaces or in other words: the solution sets of polynomial equations in the three variables x, y, and z.

How are formulas and forms connected?
The three-dimensional Euclidian space is the physical space which surrounds us and in which we move. By selecting three orthogonal coordinate axes we can fix an origin 0 and three directions x, y, and z. They make it possible for us to orientate ourselves in space by referring to them. They define the three coordinates x, y and z of a point P. These are three numbers that clearly describe and determine the position of an arbitrary point after an appropriate coordinate system has been chosen.

A formula in our case is nothing else than an equation, for example: $x^2 + y^2 = z^2$. A point lies on a figure if its coordinates satisfy the equation. The solutions are all triples (x, y, z) of numbers for which the equation holds. The thus defined points are exactly those which are located on the geometrical figure which belongs to the formula.

Herwig Hauser's forms and formulas are chosen in such a way that equations are simple and figures are plain and natural or show interesting, although very elementary, geometrical facts. Oliver Labs' images show surfaces with many singularities that are cusps which have special mathematical properties.

代数几何

代数几何处理代数，例如公式等，和几何，例如图表等之间的关系。展览展示了一些代数曲面，或者说：有三个变量x, y, z的多项式方程的解集。

公式和图表之间如何联系？我们周围的，活动其中的物理空间是三维欧几里得空间。通过选取三个互相正交的坐标轴，我们可以固定一个原点0和三个方向x,y,z。通过这些我们可以在空间中确定自己的方向。对于一个点P，它们定义了三个坐标x, y, 和z。只要选取一个合适的坐标系，这三个数字就能清晰的描述并确定任意一点的位置。

在我们的例子中一个公式就是一个方程。例如: $x^2 + y^2 = z^2$. 一个点的坐标如果满足方程，它就落在图像上。使方程成立的解都是三元数组(x, y, z)。这样定义出来的点就是公式所代表的图像上的点。

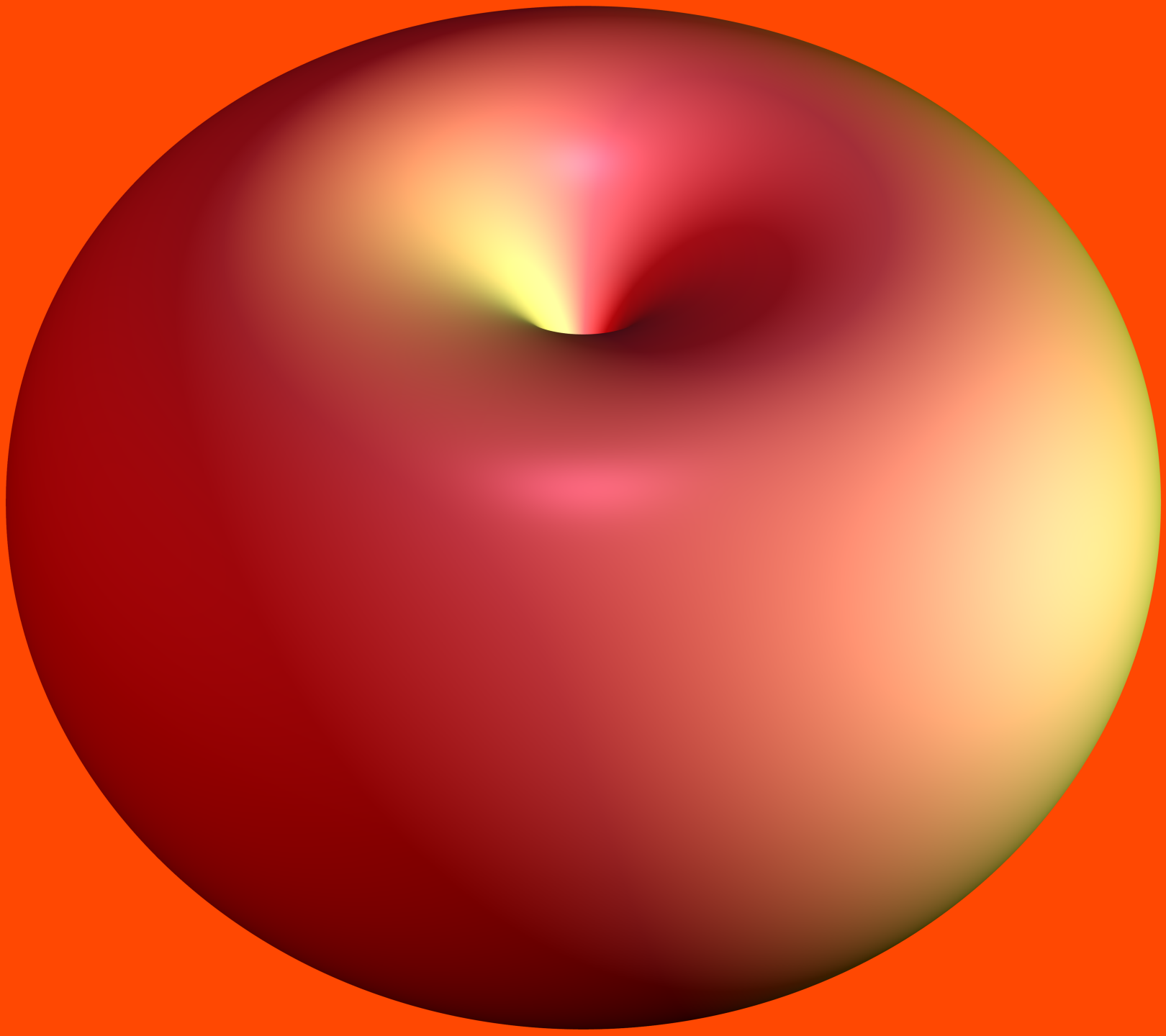
Herwig Hauser的图表和公式是选取方程简单，图像自然朴素并且能够展示基础而有趣的几何事实的。Oliver Lab的图片展示带有多个奇点尖点的曲面，这样的曲面有特殊的数学性质

DULLO

If the audience in an oval stadium scream about a score (typically of the favoured team), the sound spreads like a quickly inflated floating tyre. After some split seconds the tyre meets itself at its centre – the opening has closed – and that is what exactly happens at the kick-off spot. At that point sound waves from all directions meet simultaneously and are boosted accordingly. This is why referees are advised to always stay level with the ball. Thus, when a goal is scored they do not stand in the middle circle and get a buzzing in their ears.

决斗

如果在一个椭圆形体育场中的观众为队伍（尤其是为了自己喜欢的队伍）得分而尖叫，声音的传播就像一个快速膨胀的轮胎。几秒钟轮胎就充满了气，也就是一开始就结束了，而这就是刚开始就发生的事情。各个方向来的声波同时相遇并因而相应增强。这就是为什么裁判员被忠告要始终和球保持在同一水平。这样，当球队得分时他们才不会位于圆环的中心并因而耳朵被震得嗡嗡响。



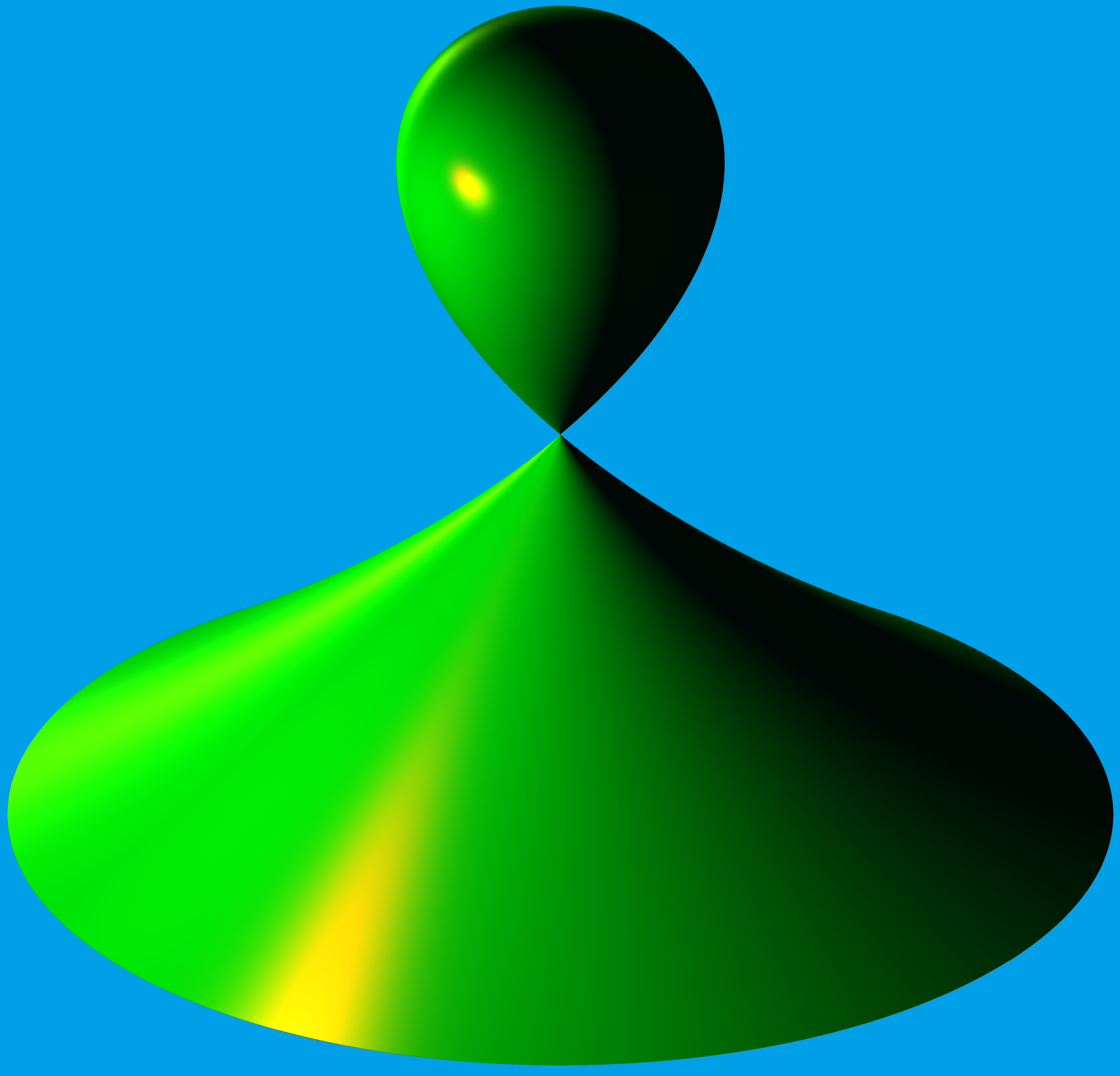
Dullo $(X^2+y^2+Z^2)^2-(X^2+Y^2)=0$

DING DONG

This surface described by the equation $x^2+y^2+z^3 = z^2$ was one of the very first visualizations we tried. Equation and shape are simple: A vertical alpha-loop rotates around the z-axis. But there was the problem with the colouring. Green is generally rather tricky in three-dimensional visualization of surfaces and, in addition, tends to be matt or yellowish. The lights and reflexions must be well tested. Note the light blue hard shadow intensifying the spatial effect.

叮咚

由方程$x^2+y^2+z^3 = z^2$所描述的这个曲面是我们最早努力可视化的成果之一。方程和形状都很简单：一个垂直的阿尔法环绕着z轴旋转。但是在涂色上存在问题。绿色通常在曲面的三维可视化中是比较厚重的，并且会倾向于缺乏光泽或者发黄。灯光或者反射波必须经过良好的测试。注意轻微的蓝色阴影会增强空间感。



Ding Dong $x^2+y^2+z^3=z^2$

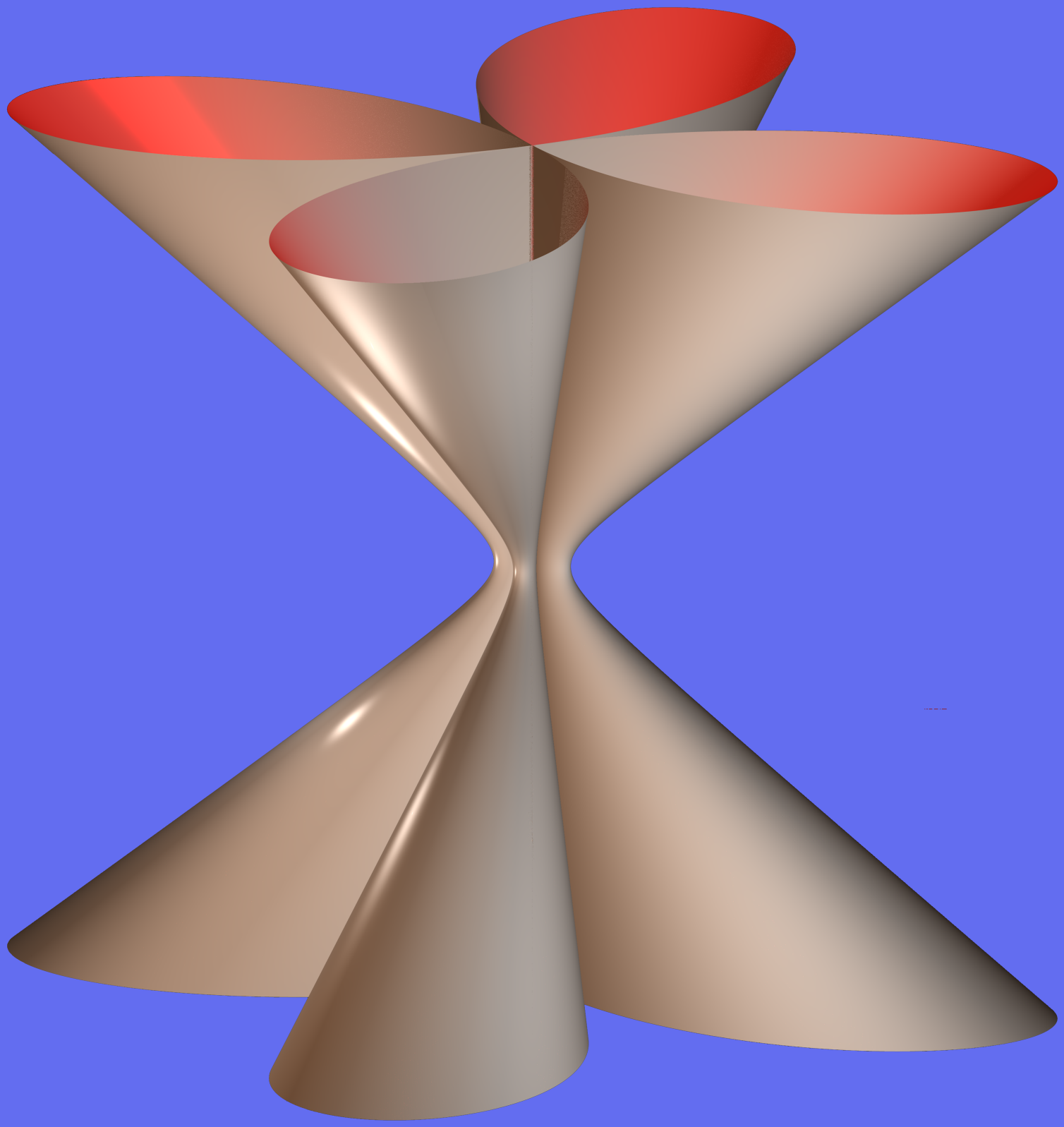
Cone

The horizontal section through Cone is a so-called rosette curve: A small wheel rolls round the interior of an annular body while a pencil fixed to the wheel is drawing a curve. Different curves are formed depending on the ratio of the two radii. The curve closes if the ratio is a rational number. In our case it is the four-leaf clover.
Our cone has the advantage that four scoops fit in. You must hurry licking otherwise the ice cream is dripping down.

锥面

锥面的水平截面被称作玫瑰型曲线：玫瑰线是由一根铅笔固定在轮子，让轮子绕环面内部旋转而画出的曲线。根据两个半径的比例不同会做出不同的曲线。如果比例是一个有理数就会做出一个闭合的曲线。在我们的例子里做出了一个四叶的三叶草。

我们的锥形的优点是嵌入了四个勺子。你必须快一点舔掉否则冰淇淋就会滴下来了。



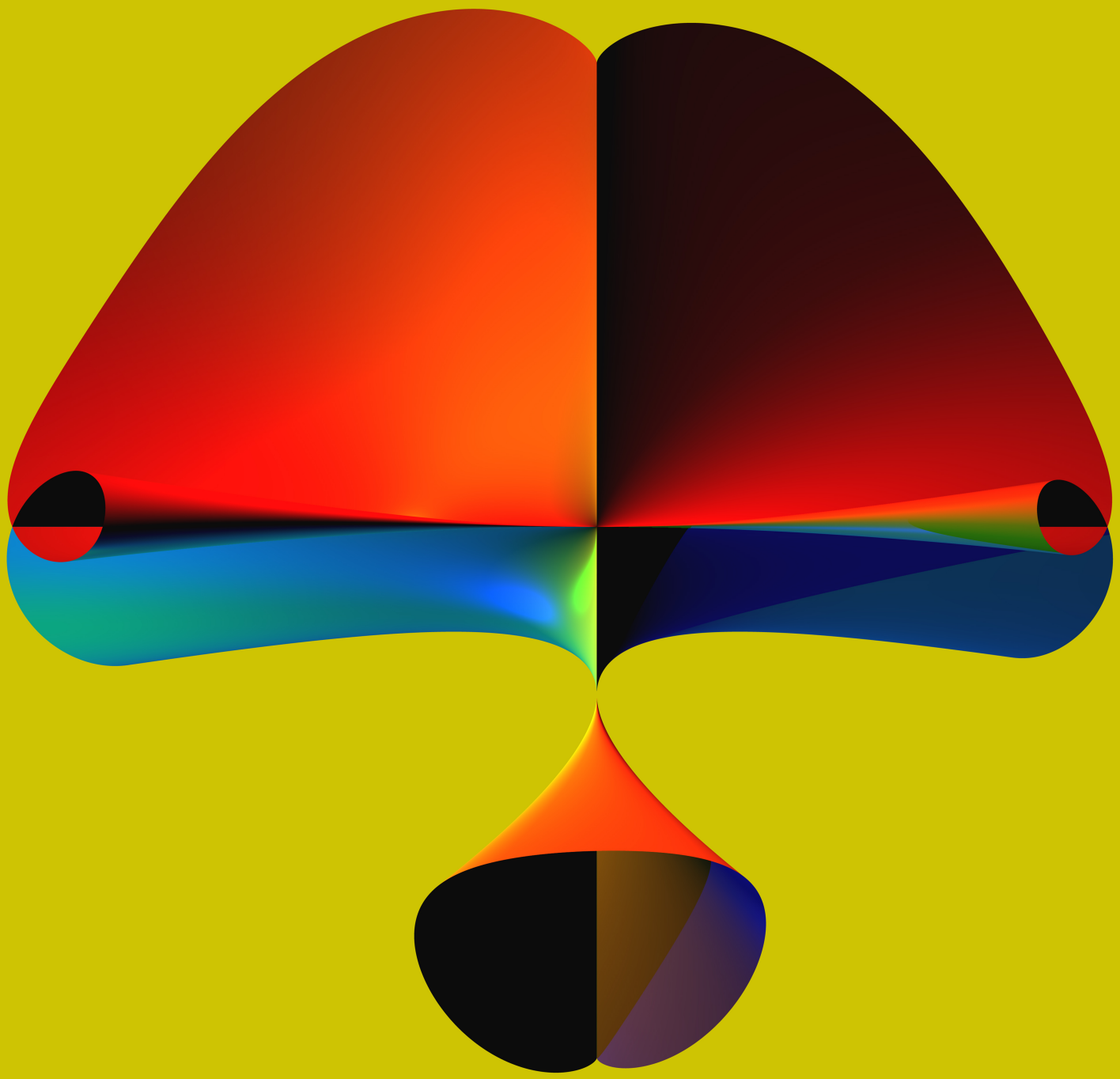
Eistute$(X^2+y^2)^3=4x^2y^2(z^2+1)$

GEISHA

This surface is, in fact, rather complicated, even if it features a regular mirror symmetry. Its name is due to its inward-turned identity: discrete, quiet and graceful. The loop-shaped bow makes one think of a kimono bow. The colours and the image result from ray tracing: A ray is traced from each point in the image and where it hits an object the colour value is calculated by means of light source models. The grace of Geisha is underlined by special lighting.

艺妓曲面

艺妓曲面是整齐的镜面对称的，但事实上它并不简单。它的名字是源于它的内旋特征：离散，安静并且优美。环形的鞠躬让人想起和服鞠躬礼。色彩和图片来自于光线追踪：一根光线从图片上的一个点开始追踪，当它遇到一个物体时，就用它的光源模型计算出色彩值。特别的照明更加强调了这个曲线的优美。



Geisha $x^2yz+x^2z^2=y^3z+y^3$

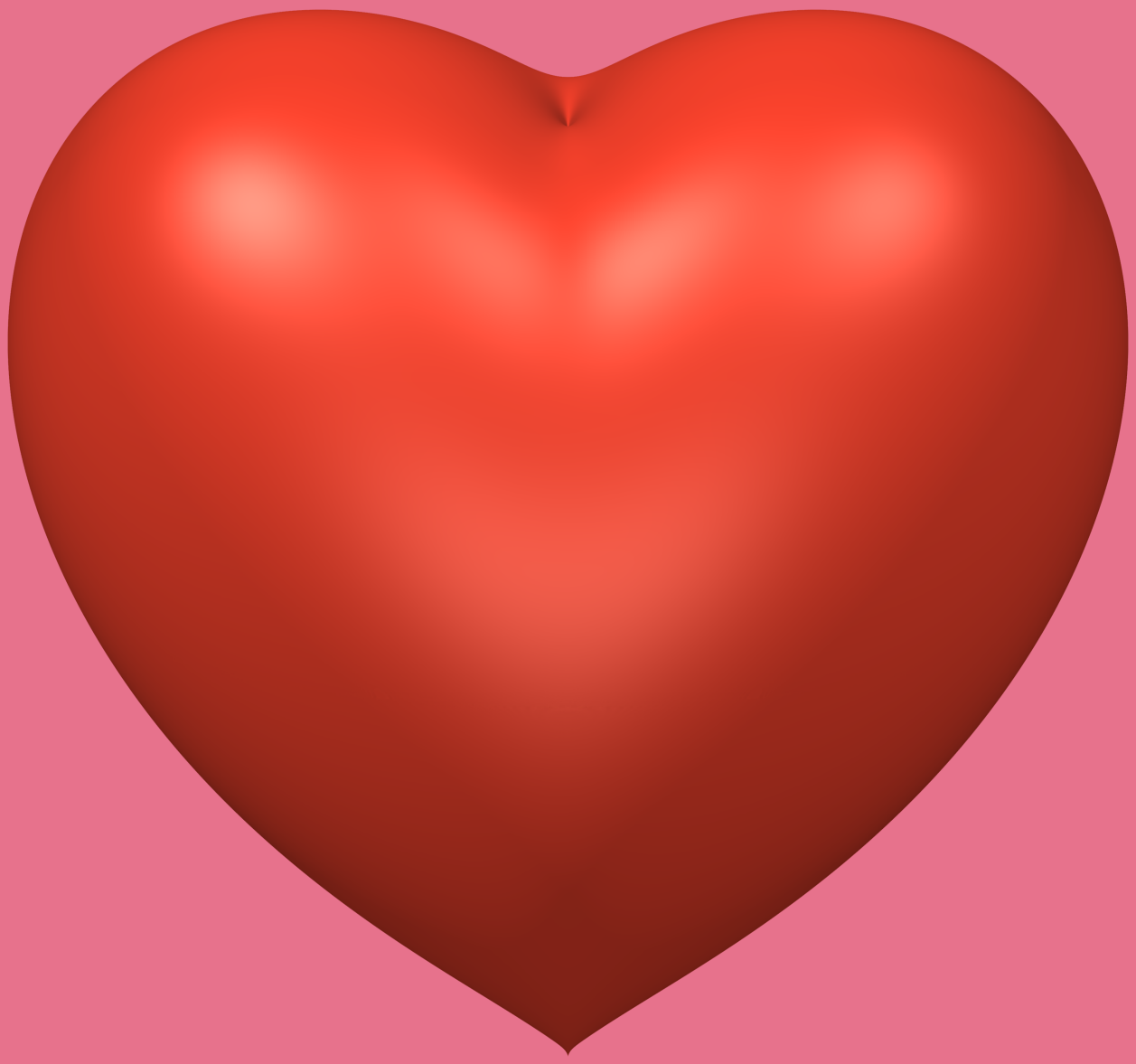
Design: Sebastian Gann, Christian Stussak

SWEET

Notes from algebraic geometry: Vertices or singularities are very interesting from the mathematical point of view. These points behave extraordinarily. Tiny changes in the equation have strong effects on these points.

甜心曲面

来自代数几何：从数学角度来看顶点和奇点是非常有趣的。这些点行为异常。方程中微小的改变可以对这些点产生巨大的影响。



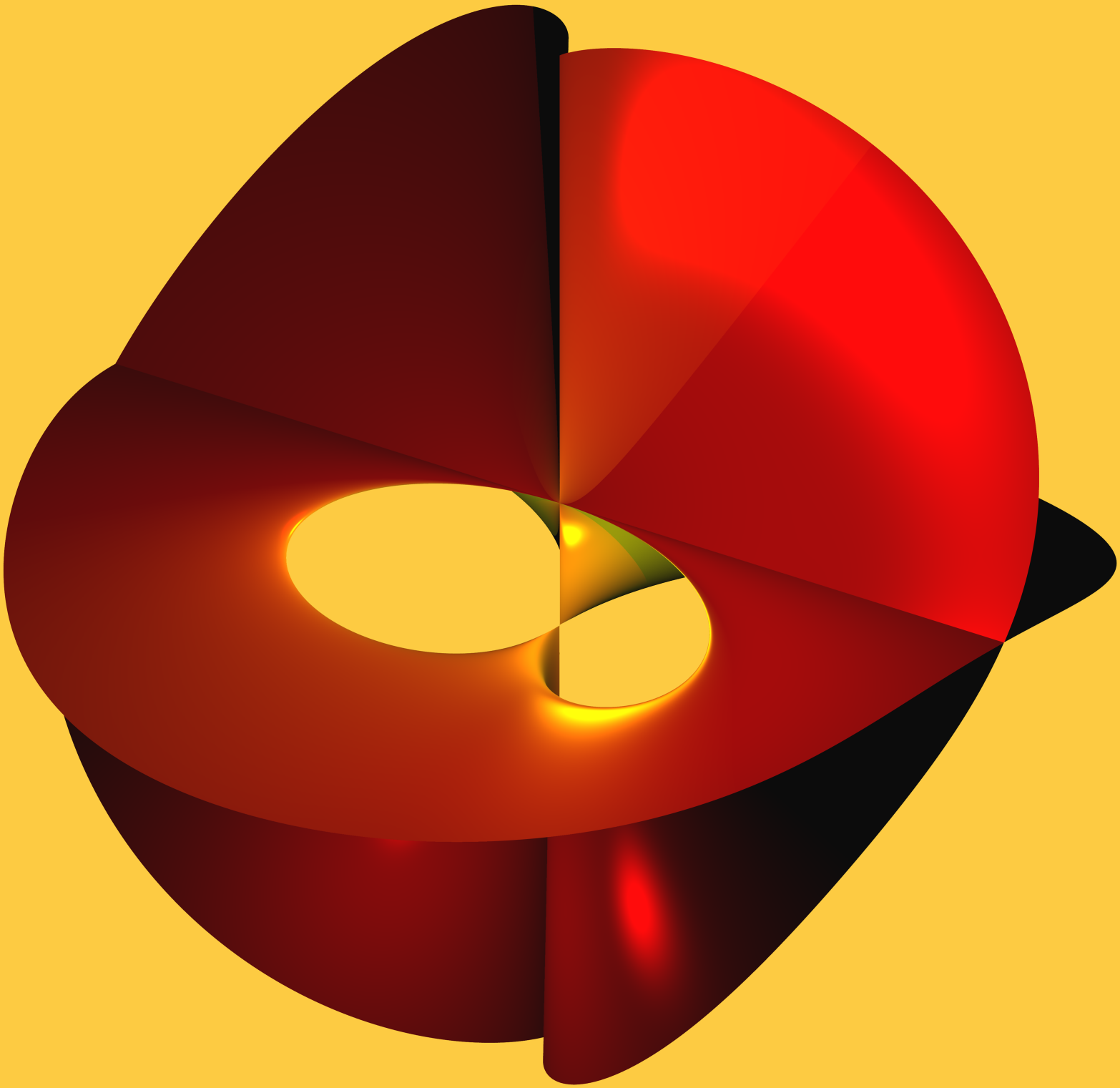
Suss $(x^2+9/4y^2+z^2-1)^3-x^2z^3-9/80y^2z^3=0$

MIAU

This surface was developed by coincidence on a drab train ride (Working with algebraic visualization makes time elapse quite fast). But the constellation isn't a coincidence at all, on the contrary, if one wanted to systematically derive the algebraic equation for this surface one would face unsurmountable problems. The challenge of Miau is, of course, the double opening with embedded singularity. For mathematicians this is a treasure trove exploring the relationship between equation and form.

妙悟曲面

这个曲面是在一次乏味的火车旅行中偶然发现的（做代数可视化工作，时间消耗得很快）. 但是这个系列的发现并不完全是偶然的，相反，如果想系统地导出这个曲面的代数方程将面临无法克服困难。当然妙悟曲面面临更困难的挑战，在于其双开口外的嵌入奇点上。对于数学家来讲，探索方程和形式（方程对应的曲面）之间的关系就像一个寻宝的过程。



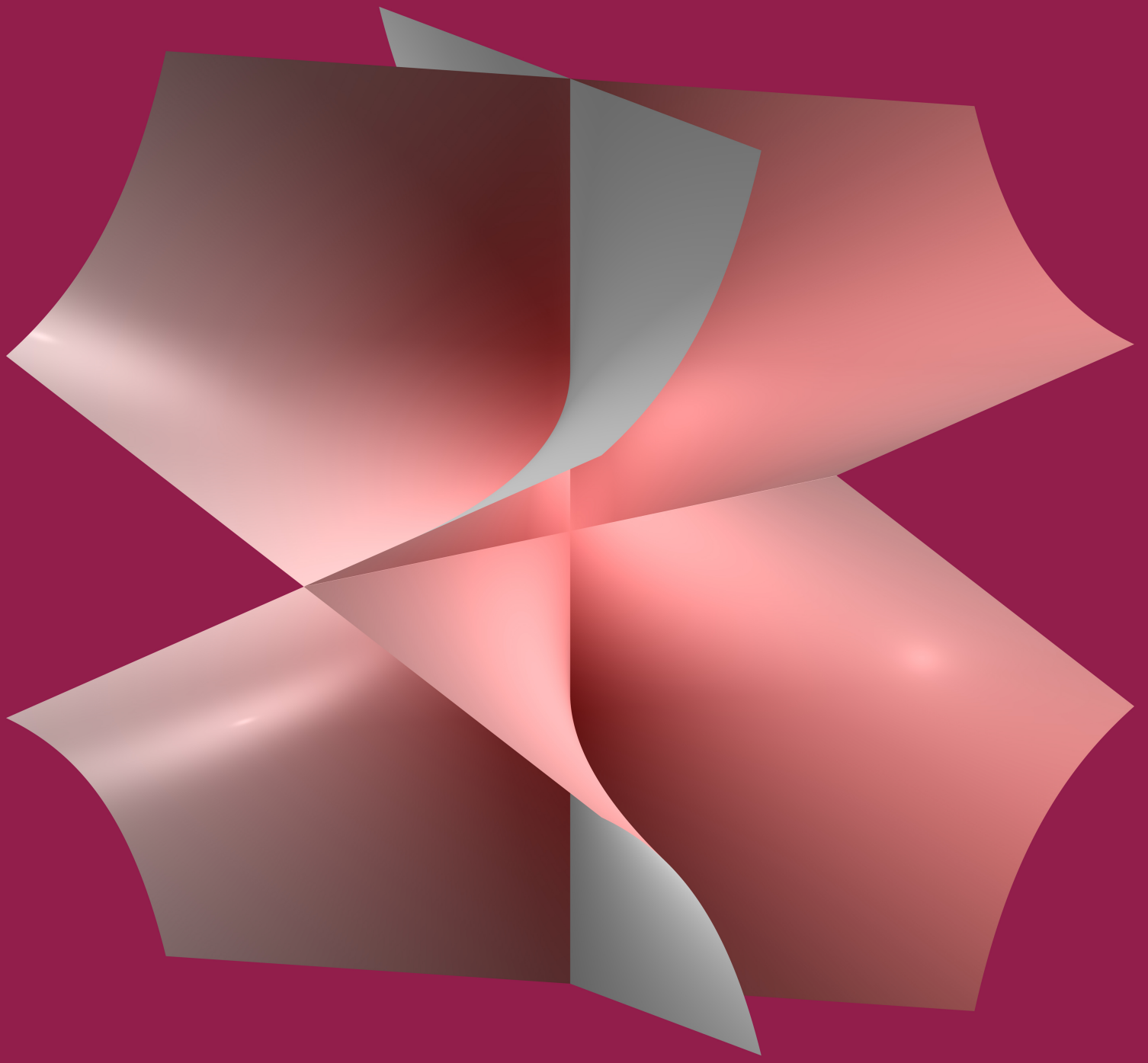
Miau $x^2yz+x^2z^2+2y^3z+3y^3=0$

HEAVEN AND HELL

A piece of paper is folded and is held from beneath such that you can put your four fingers in the four corners so formed. By spreading your fingers the figure opens in two different ways so that two of the four inner sides can be seen at a time, the blue ones for heaven, the red ones for hell. Children guess which colour will show up. Our figure reminds us of this game, hence the name.
By adding up the squares at y and z you get the highest exponent 4. This is called an equation of the 4th degree. The higher the degree the more complicated it is to calculate the surface.

天堂与地狱

曲面的名称来自于一个儿童猜色游戏。把一张纸折叠然后把你的四个手指放在四个角从下面捏紧就会形成这样一个曲面。伸开你的手指，曲面会由两个不同方式展开，可以同时看到四个内面中的两个，蓝色的表示天堂，红色的表示地狱。将y的次数和z的次数相加得到方程的最高阶数为4阶，所以这个方程称为四阶方程。方程的阶数越高推断曲面就会越复杂。



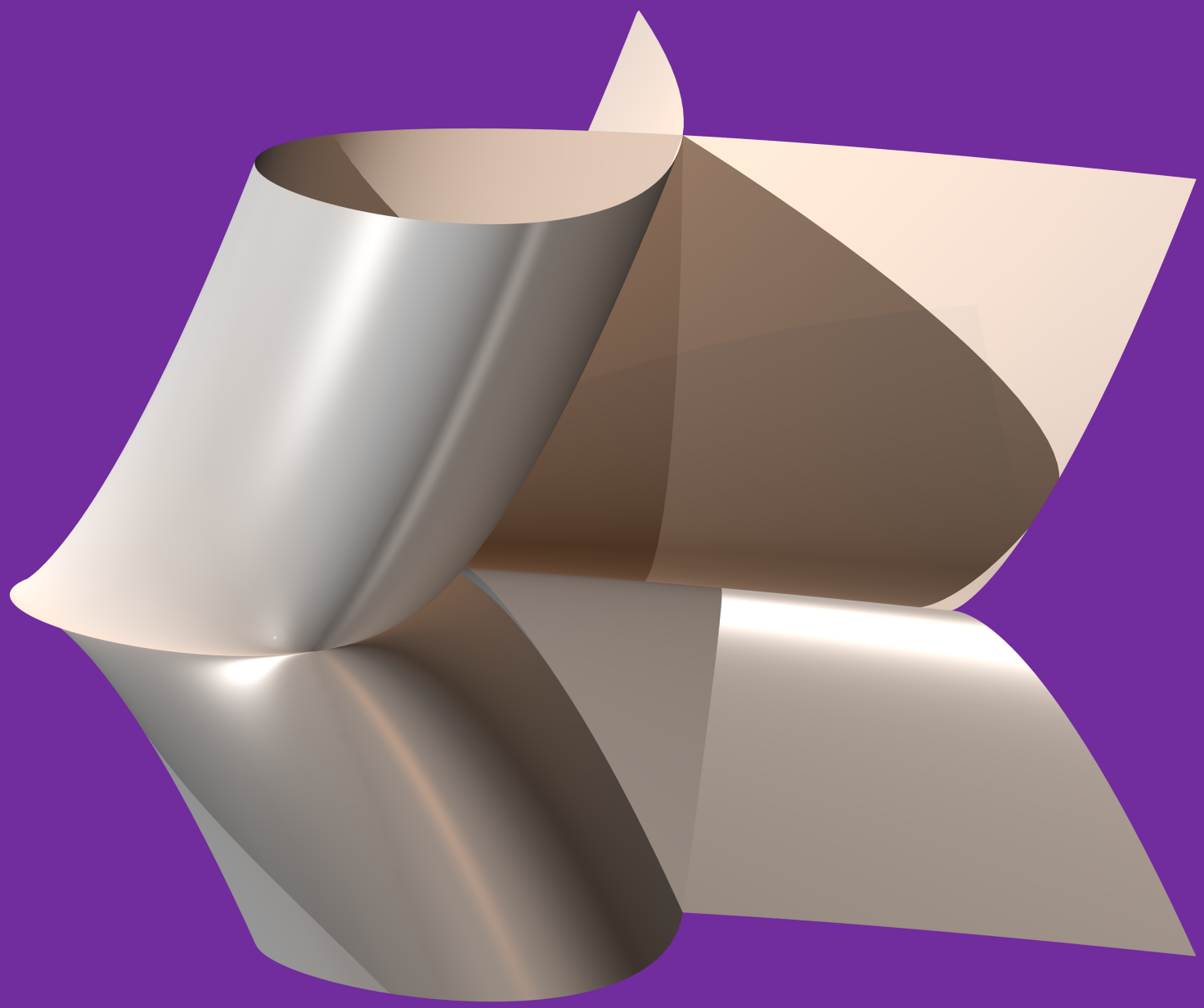
Himmel und Holle $x^2-y^2z^2=0$

QUASTE

The upper border of this surface is a loop in the shape of the Greek letter Alpha, whereas the border to the right consists of two parallel so-called cuspidal curves with a spike each. By moving the horizontal loop downwards, leading its two endpoints on the right along the curve, the surface Quaste will come out. On the other hand, you can also shift one of the curves along the loop and you will get the same figure. Surfaces with this property are named Cartesian products after the French mathematician René Descartes.

凯斯特曲面

曲面的上边界是一个像希腊字母α的环形，而右边界由两个具有尖点的所谓尖点曲线组成。向下移动水平环，同时带领其右侧的两个端点沿曲线移动，就可以形成凯斯特曲面。 另一方面，你还可以沿着环移动其中的一个曲线，从而得到一个相同的图形。为了纪念具法国数学家勒内·笛卡尔，这个曲面也被被称为笛卡尔乘积。



Quaste

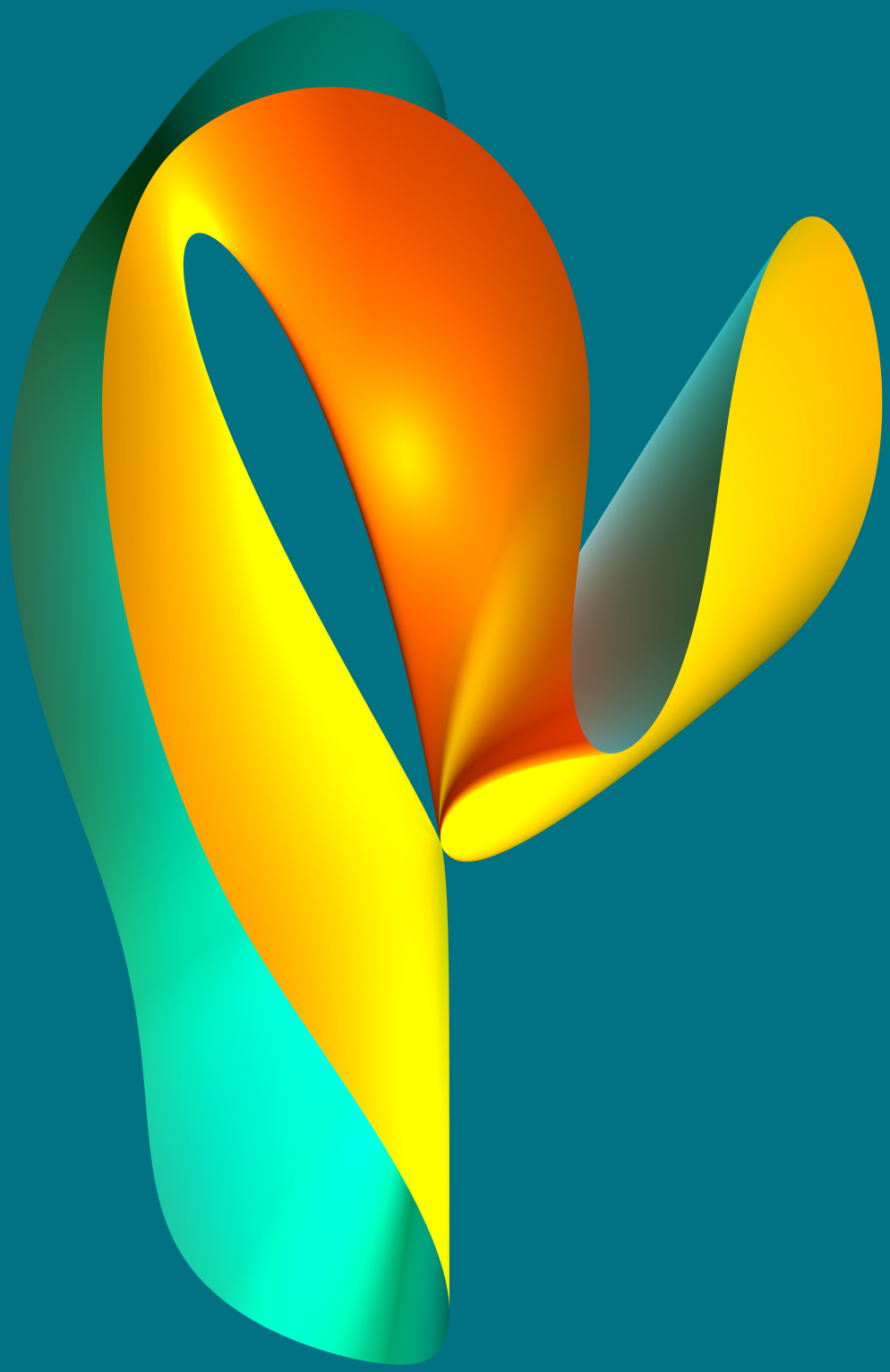
$$8z^9-24x^2z^6-24y^2z^6+36z^8+24x^4z^3-168x^2y^2z^3+24y^4z^3-72x^2z^5-72y^2z^5+54z^7-8x^6-24x^4y2^-24x^2y^4-8y^6+36x^4z^2-252x^2y^2z^2+36y^4z^2-54x^2z^4-108y^2z^4+27z^6-108x^2y^2z+54y^4z-54y^2z^3+27y^4=0$$

SEAHORSE

If you want to find the equation of this surface it would take strong efforts. The soft tangential contact is not easy to achieve. It vanishes as you only slightly change the formula.
The elegance of the sea horse is an illusion: If you look at it from behind or from the side, it appears quite clumsy. Sea horses live worldwide in tropical and temperate climate zones. Its Latin name is Hippocampus, you can find the equation next to it.

海马曲面

找到这个曲面的方程非常困难。软切向接触并不容易得到。你只要稍微改变公式，它就会消失。优雅的海马是一个错觉：从后面或从侧面看上去，它显得很笨拙。海马广泛地生活在世界各地的热带和温带气候区。你可以在方程旁边找到它的拉丁名称Hippocampus.



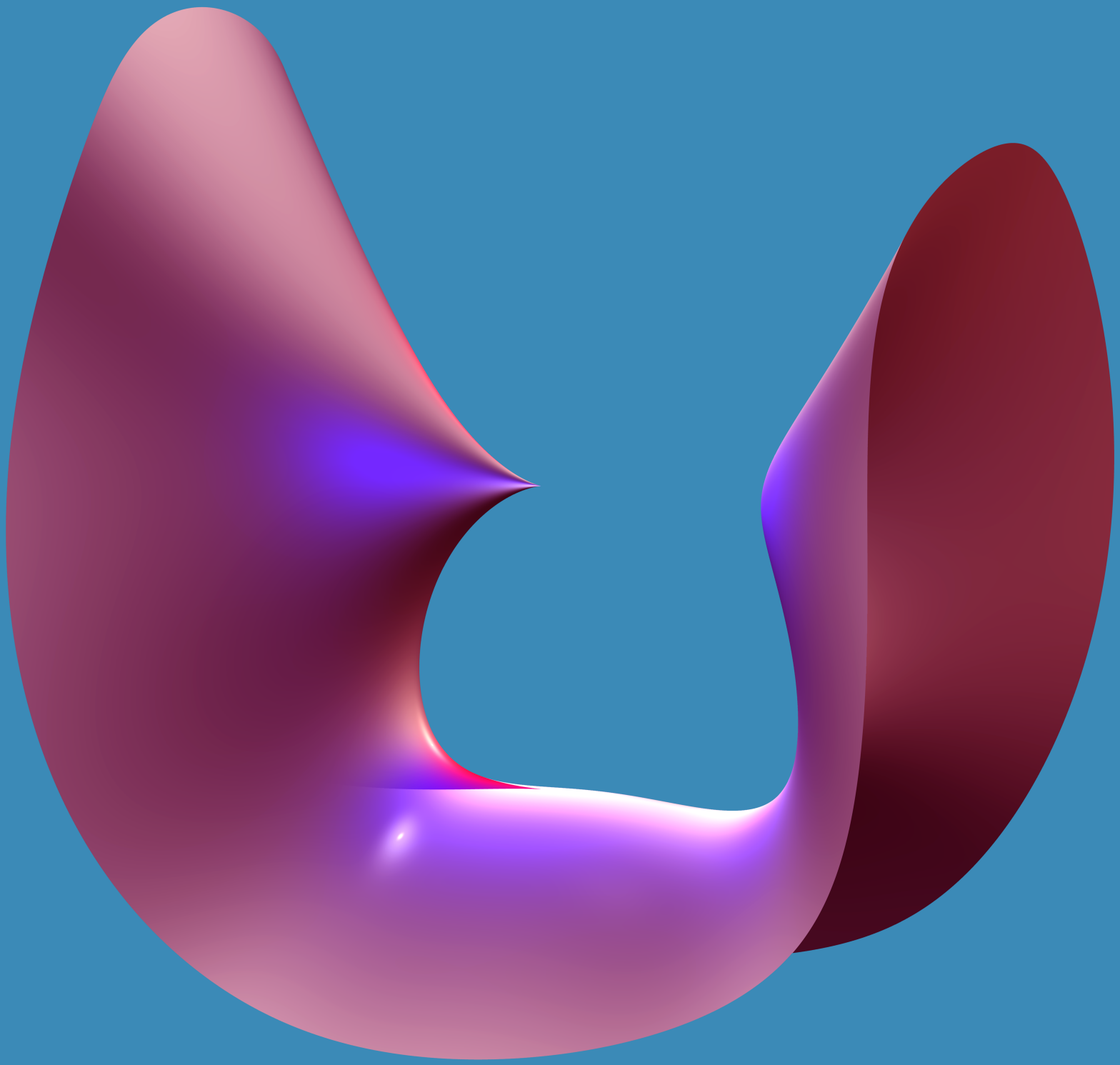
Seepferdchen $(X^2-y^3)^2=(X+y^2)z^3$

VIS À VIS

Vis à Vis means opposite – and, here, two essential phenomena of algebraic geometry stand opposite each other. The singular tip on the left looks at a curved but smooth hill on the right. This singularity is more exciting, because various changes to the equation can result in unpredictable changes to the figure, which does not happen at smooth points. By using the SURFER program, which is available for free at surfer.imaginary2008.de such surfaces can be generated and modified quite easily and intuitively. The comparison of form and formula, i.e. of equation and corresponding surface, becomes an interactive experience which is intriguing to understand.

面对面

Vis à Vis的意思是面对面，体现了代数几何中的两个互相对立的基本现象。从左边看，可以看到奇异点，望向右边可以看到弯曲而光滑的山。这个奇点非常令人兴奋，因为方程的不同变动会导致图形发生不可预测的变化，这在光滑点是不可能发生的。我们可以利用surfer.imaginary2008.de上的免费程序SUFFER，非常容易和直观地生成或改变这个曲面。在这个软件中，比较曲线与公式，或者说方程和对应的曲面，会有一种引人入胜的交互式的体验。



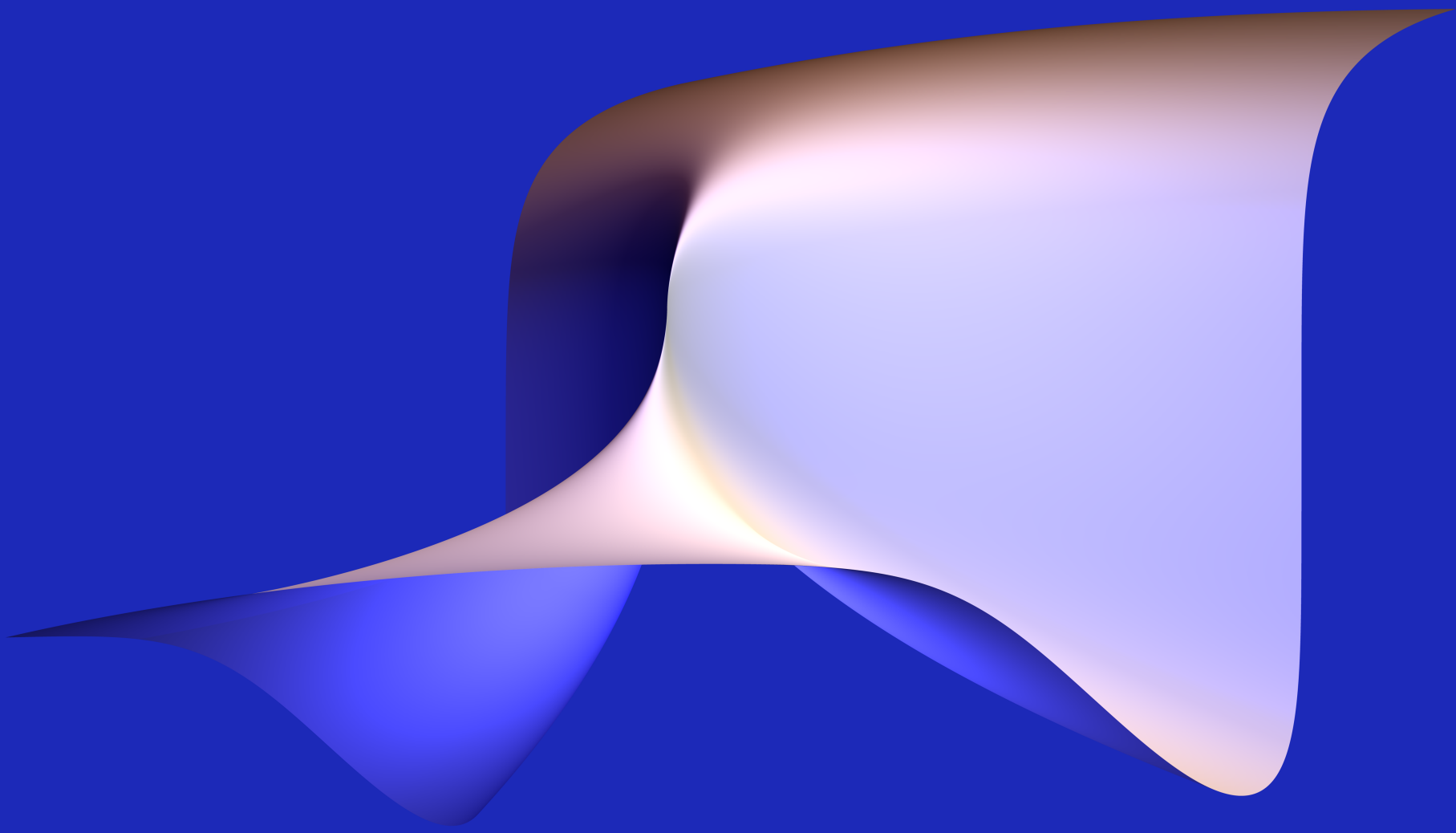
Visavis $X^2-x^3+y^2+y^4+z^3-z^4=0$

SOFA

Though an algebraic surface is called “Sofa” sitting on it is not necessarily comfortable. What we have here, rather, is a seating for two separated from each other by a singularity. This singularity is called E8 in mathematical language and is perhaps the most famous of all singularities. It combines, among others, the theory of the symmetry groups of Platonic solids (E8 is part of the icosahedral symmetry group) and the theory of the Lie groups. The real image of this singularity excels through its elegance, though it does not reveal its mathematical complexity, which only becomes apparent as you include the imaginary part.

沙发曲面

虽然这个代数曲面称为“沙发”，坐在它上未必舒适。相反，这里我们面对的是一个由奇点分割开的两个座位。这个在数学上称为E8奇点也许是所有奇点中最著名的一个了。它还结合了柏拉图立体的对称群理论(E8是二十面体对称群的一部分)和Lie群理论.虽然你仍无法揭示数学的复杂性，但你可以凭借想象力清晰地想象出其真实情况。



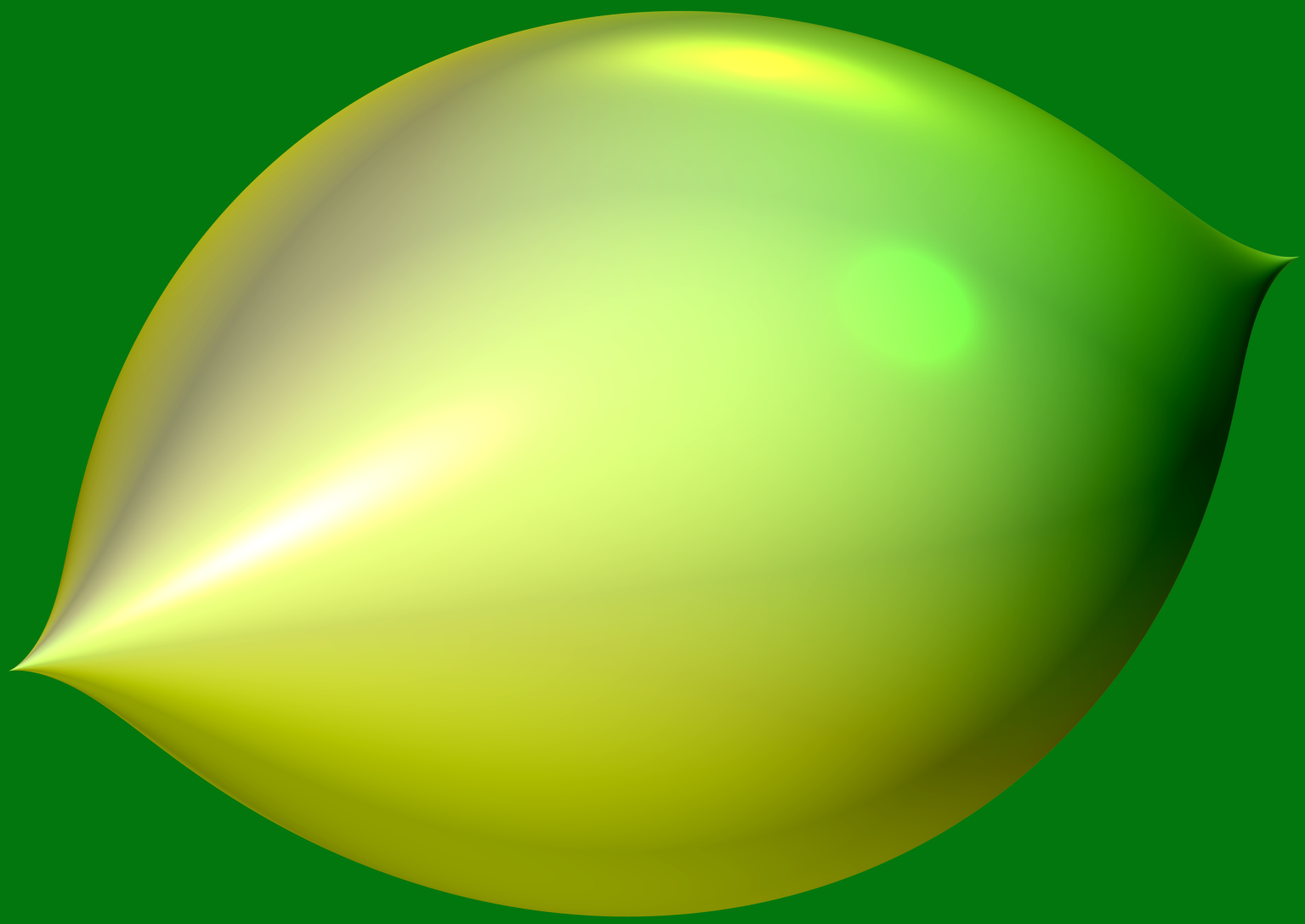
Sofa $x^2+y^3+z^5=0$

CITRIC

The equation $x^2+z^2 = y^3(1-y)^3$ of Citric appears as simple as the figure itself. Two cusps mirror-symmetrically arranged rotate around the traversing axis. The equation $x^2+z^2 = y^3$ simplified by omitting $(1-y)^3$ provides for exactly one cusp, and $x^2+z^2 = (1-y)^3$ yields the mirror image. Both are infinitely extending surfaces. The product on the right side of the initial equation ensures that Citric remains bounded. You may consider the following: If the absolute value of y is getting larger than 1 the right side becomes negative and the equation does not admit real solutions of x and z.

柠檬曲面

柠檬的方程$x^2+z^2 = y^3(1-y)^3$和它的图像一样简单。两个尖点绕其对称轴旋转成镜面对称，就会得到柠檬曲面。等号右侧省略掉$(1-y)^3$后得到$x^2+z^2 = y^3$，这个方程表示曲面只有一个尖点，另一个方程$x^2+z^2= (1-y)^3$产生其镜面对称图像。两个曲面都是无限延伸的曲面。将这两个初始方程的右侧相乘保证柠檬曲面有界。可以考虑以下问题：如果y的绝对值大于1，右侧的符号为负，则方程没有x,z的实数解。



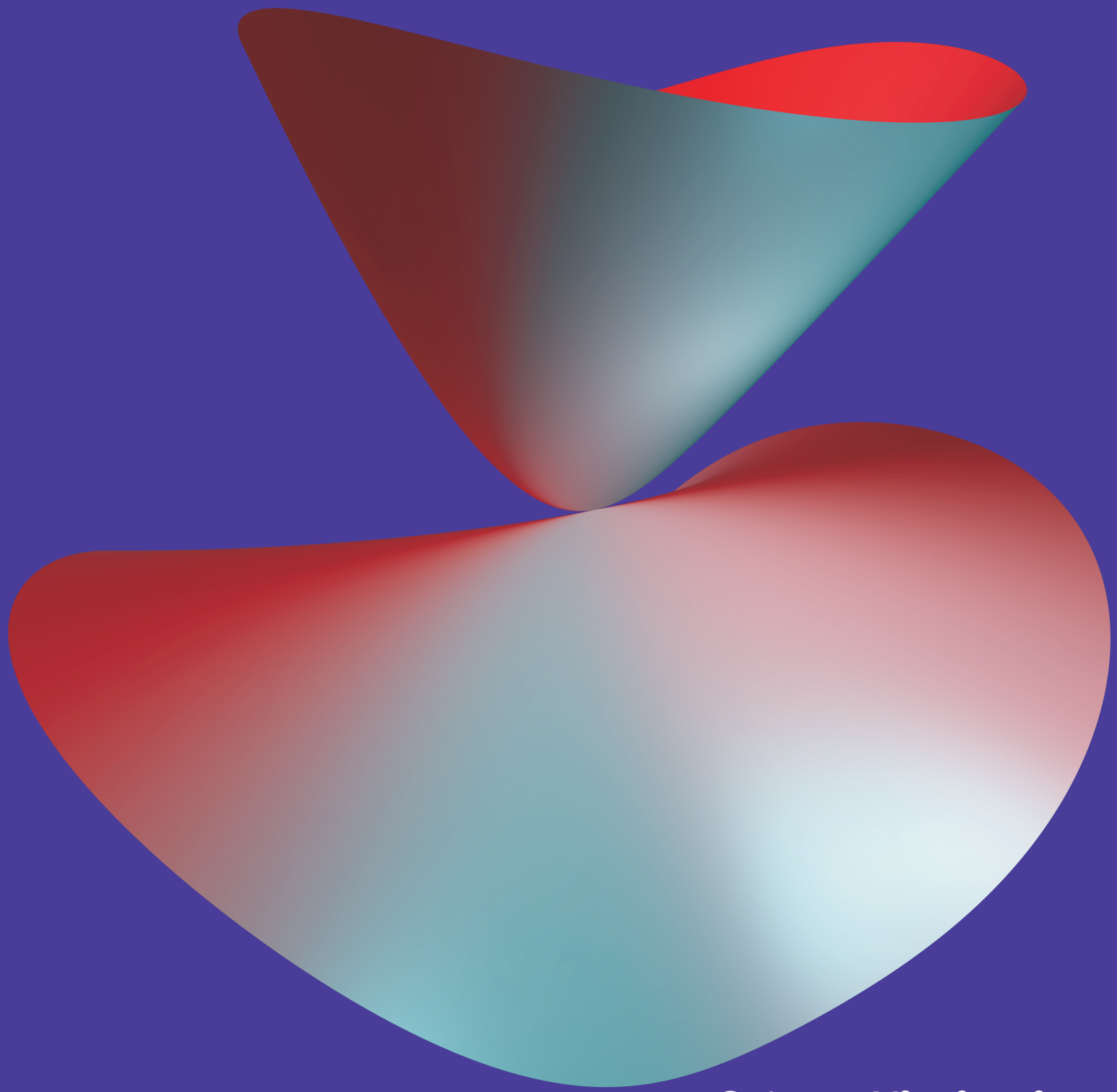
Zitrus $x^2+z^2=y^3(1-7)^3$

CALYPSO

The surface Calypso with equation $x^2+y^2z = z^2$ contains three straight lines. The horizontal straight line is clearly visible, it passes through the origin (zero) where the upper and lower part meet. The two other straight lines lie in a vertical plane, they also pass through 0 and intersect each other at that point. The section of the surface with this plane shows the two straight lines.
If you shift this plane a bit forward the section curve turns into a hyperbola. This can be easily checked by calculation. You set either $y=0$ or $y=1$. In the first case the result is $x^2 = z^2$ or $(x-z)(x + z) = 0$, the equation of two straight lines in the plane.
In the second case you get $x^2+z = z^2$. This can be rewritten in $-x^2+(z-1/2)^2 = 1/4$, the equation of a hyperbola with centre at (0, 1/2).

卡利普索曲面

方程$x^2+y^2z = z^2$对应的Calypso曲面包含3条直线。其中水平的直线是清晰可见的，它通过上部和下部的交汇处——原点。其他两条直线在垂直面内，他们也通过原点，并在该点彼此相交。曲面和这个垂直面相交的交线为两条直线。如果把这一平面稍微向前移动，交线就会变成双曲线。这个结果可以很容易验证。设$y=0$或$y=1$.在第一种情况下，我们得到方程$x^2 = z^2$或 $(x-z)(x + z) = 0$，这是xOz平面内的两条直线方程。在第二种情况下，我们得到方程$x^2+z = z^2$，将该方程改写为$-x^2+(z-1/2)^2 = 1/4$，这恰是一个以(0, 1/2)为中心的双曲线。



Calypso $X^2+y^2z=z^2$

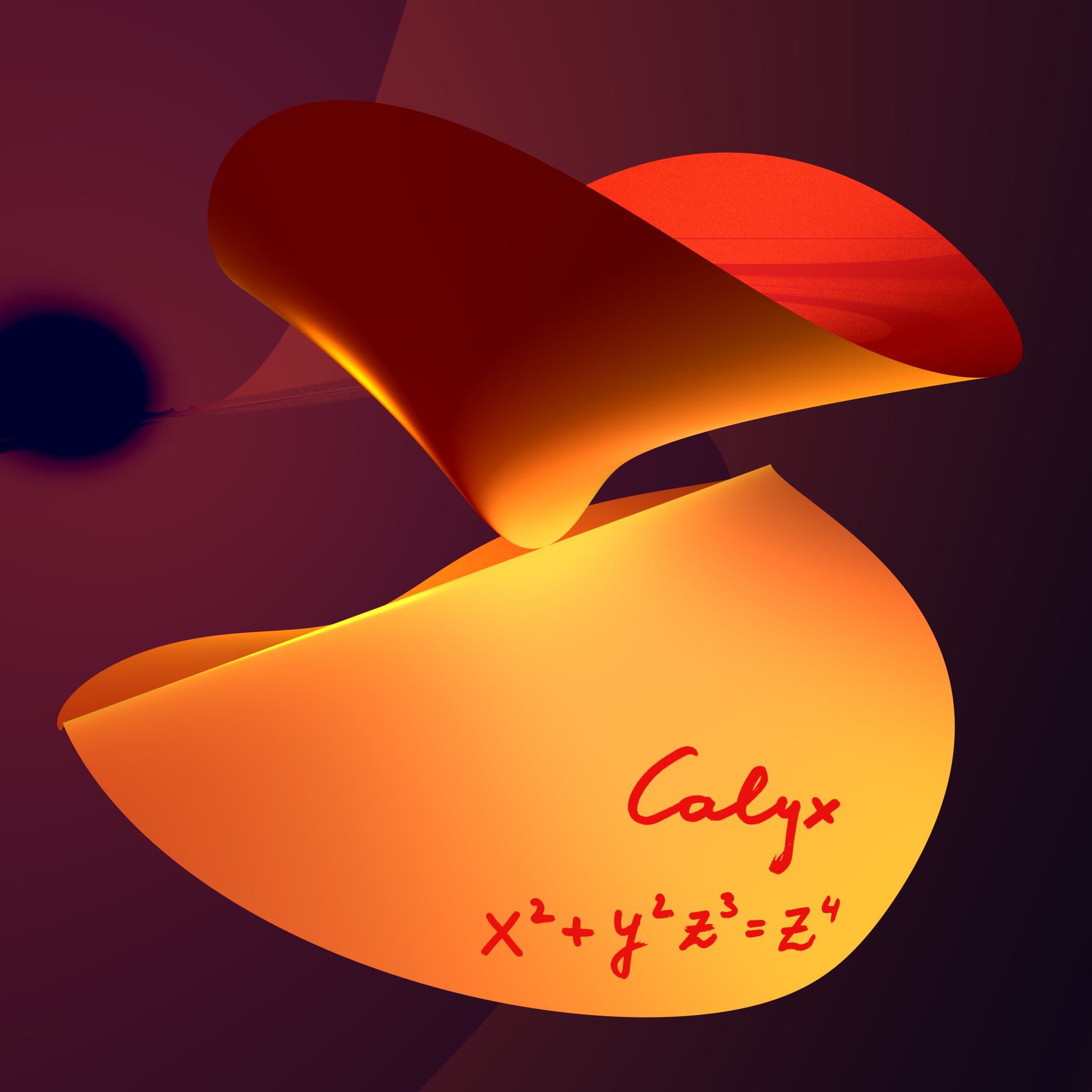
Design: Herwig Hauser, Christian Stussak

CALYX

The surface Calyx with equation $x^2+y^2z^3 = z^4$ has a straight line as its singular locus. The lower part of the surface has cusp-shaped singularities alongside the straight line, whereas the upper part tangentially touches the straight line at a point, the origin. The real image is misleading insofar as the defining polynomial is irreducible and the surface, as a result, consists just of one algebraic component (not of two components as the figure suggests).
It can be shown that Calyx is an appropriate projection of Calypso. In a three- dimensional space a cylinder surface is contracted to the singular straight line of Calyx. Algebraically, the mapping is defined by the requirement $(x, y, z) \rightarrow (xz, y, z)$. It is very simple. The respective substitution in $x^2+y^2z^3 = z^4$ and subsequent reducing of z^2 yields the Calypso equation $x^2+y^2z = z^2$.

花萼曲面

方程$x^2+y^2z^3 = z^4$表示的曲面——花萼曲面有一条直线的奇异轨迹。曲面的下半部分沿着这条直线有尖角形奇异点，而曲面的上半部分在原点与这条直线垂直相切。因为定义曲面的多项式是不可约的，这个图像有时候会有误导作用。且仅包含其中一个代数组成（而不是图中应有的两个部分）。可以证明Calyx是Calypso的投影。在三维空间，圆柱体表面收缩为Calyx的奇异直线。曲面从代数上说，这是非常简单的映射：$(x, y, z) \rightarrow (xz, y, z)$。在方程$x^2+y^2z^3 = z^4$中将对应变量替换为映射右侧的对应变量，即x替换为xz，y、z不变，并约去z2就得到Calypso 方程 $x^2+y^2z = z^2$.

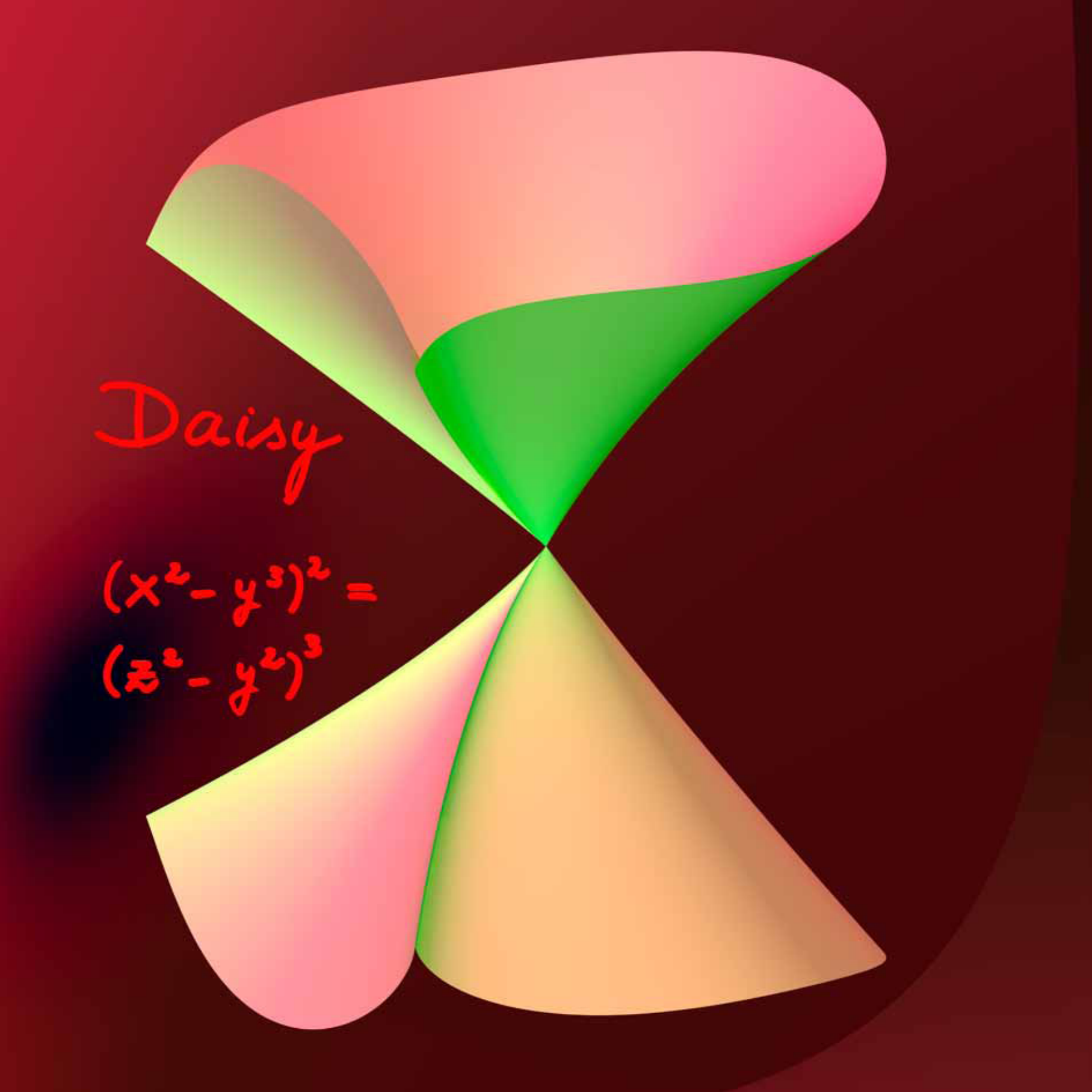

Calyx
$x^2 + y^2 z^3 = z^4$

DAISY

The equation $(x^2-y^3)^2 = (z^2-y^2)^3$ of Daisy implies by differentiation that the singular locus consists of two (plane) curves wh transversally meet at their common singular point. In order to better understand singularities the geometrician constructs their *resolution* by means of *blowups*. In finitely many steps they provide a surface without singularities (a manifold) together with a projection map onto the original surface which interprets it as a shade of manifold.

雏菊

通过求导可知，雏菊方程$(X^2-y^2)^2 = (z^2-y^2)^3$奇异点的轨迹是由两条（平面）曲线组成，这两条曲线横向相交于它们公共的奇异点。 为了更好地理解奇异点，几何学家通过胀开的方式来构造它们的分解。 有限步以后，他们得到了一个无奇点的曲面（流形）和映满到原曲面的一个投影映射。原曲面可以理解为这个流形的底影。

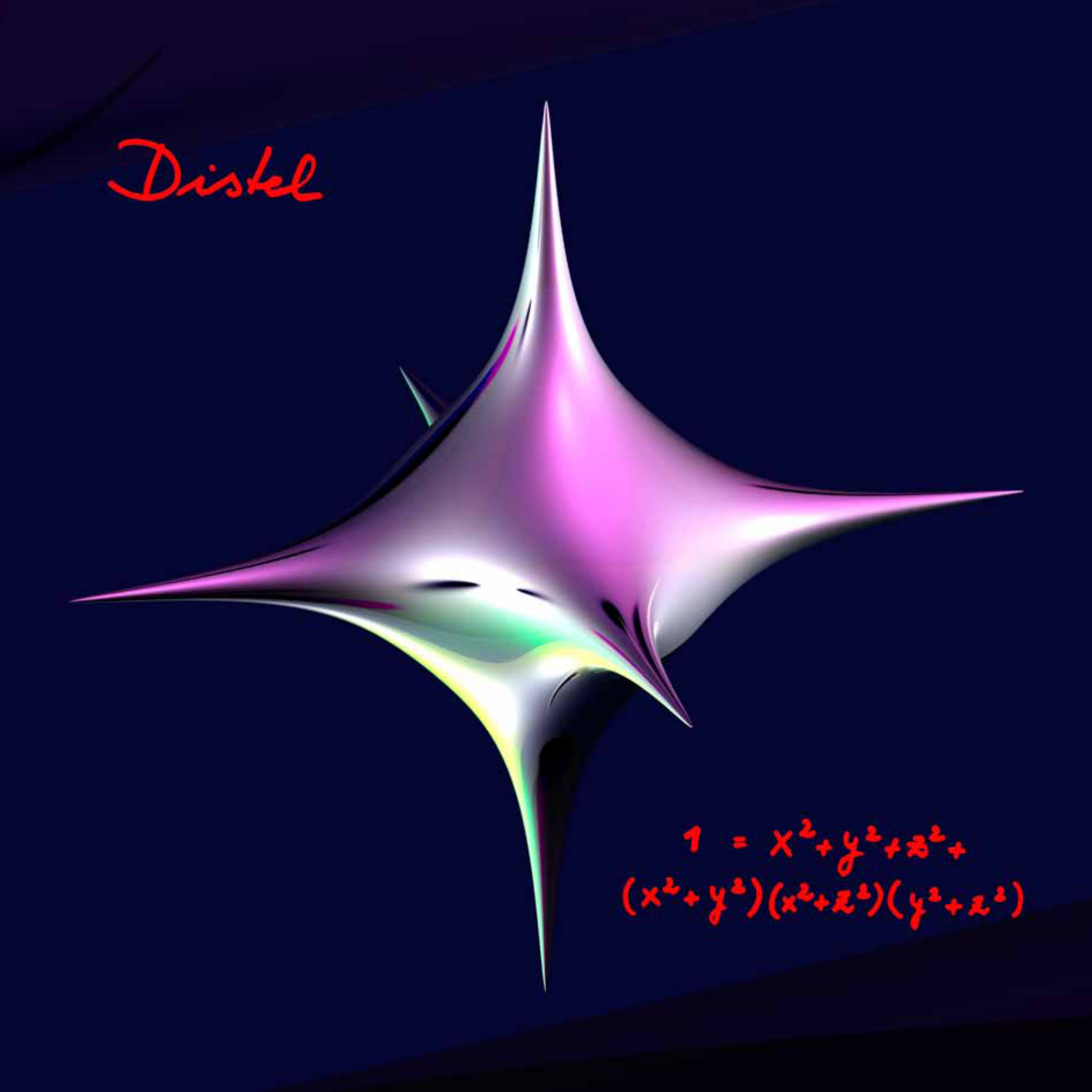

Daisy
$(x^2 - y^3)^2 =$
$(z^2 - y^2)^3$

THISTLE

The surface Thistle with the equation $x^2+y^2+z^2+c(x^2+y^2)(x^2+z^2)(y^2+z^2) = 1$ excels through its extraordinary symmetry. The real image was provided with a very big coefficient c. The six spikes are located on the three coordinate axes of the Euclidian space. Each twist permuting the three axes leaves Thistle unchanged. The symmetry group, as a result, is that of the cube and the octahedron which is dual to it, two of the five Platonic solids.
Surprisingly, it is not possible to construct totally regular stars such as Thistle having any number of spikes. This results from considerations of group theory. There can be only four, six, eight, twelve or twenty spikes according to the sides of the Platonic solids. We leave it to the curious viewer to find the appropriate equations for all these stars.

THISTLE

由方程$x^2+y^2+z^2+c(x^2+y^2)(x^2+z^2)(y^2+z^2)=1$表示的Thistle曲面具有非常好的对称性。下面提供的实图对应的方程带有一个非常大的系数c。六个尖峰位于欧氏空间的三个坐标轴上。任意置换三个坐标轴的变换保持Thistle曲面不变。Thistle曲面的对称群（也叫自同构群）与立方体和八面体 的对称群同构。立方体和正八面体互为对偶，属于五个柏拉图体（正四面体，立方体，正八面体，正十二面体和正二十面体）中的两个。 令人惊讶的是，基于群论的角度考虑，不能构造出像Thistle曲面一样带有任意个尖峰的完正则的星形。 类似柏拉图体，只可能存在四，六，八，十二和二十个尖峰。我们把它留给好奇的读者去寻找这些星形适合的方程。


Distel
$1 = x^2+y^2+z^2+$
$(x^2+y^2)(x^2+z^2)(y^2+z^2)$

HELIX

The Lemniscate is the plane curve with the equation $y^4+z^2 = y^2$. It results from the circle $y^2+z^2 = 1$ by substituting z by z/y. Geometrically, the substitution corresponds to the one-time torsion of the circle to a figure-eight loop. If $y^4+z^2 = y^2$ is conceived as equation with three variables x, y and z (where x is a stowaway) then the solution set is a surface in the three-dimensional space, i.e. the cylinder above the lemniscate.
The Helix equation $y^4+x^2z^2 = y^2$ is yielded by substituting z by xz. From the geometrical view, this construction is a kind of folding. The symmetry with respect to x and z is clearly visible. For the final formula we added the factors 2 and 6 to slightly stretch Helix. The singular locus is a cross of straight lines. The sections of Helix with the planes $x = c$ or $z = c$ are lemniscates for $c \neq 0$, whereas the sections $y = c$ are hyperbola pairs.

双纽线

双纽线是一条平面曲线，其方程为$y^4+z^2=y^2$。它是在圆的方程$y^2+z^2=1$中将z/y替换z而得到。几何上，这个代换对应于圆到‘8’字环的一次扭转。如果将$y^4+z^2=y^2$看作是关于x,y,z的方程（其中x省略），那么它的解集就是三维空间中的曲面，即双纽线上的柱面。在双纽线方程中将xz替换 z即得螺旋线方程$y^4+x^2z^2=y^2$。从几何上看，这个构造可以看成是一个折叠。关于x和z的对称性是显而易见的。为了得到最后的公式，我们增加因子2和6来轻微地拉伸螺旋线，奇异点的轨迹是两条交叉的直线。当$c\neq0$时，平面x=c和平面z=c与螺旋面相截为双纽线，平面y=c与螺旋面相截为双曲线对。

Helix
$6x^2 = 2x^4 + y^2z^2$

HEART

Despite the simple equation $y^2+z^3 = z^4+x^2z^2$ the surface Heart possesses a subtle local and global structure. The singular locus is a straight line alongside which the surface intersects itself. The origin 0 is the interesting point: We intersect Heart with planes $x = c$ orthogonal to the singular straight line. The result is a loop which contracts like a knot if c tends to 0. A funnel is formed. Viewed from the distance we see a circle shaped opening in the surface. The section with the vertical xy-plane is in fact a circle. The simplicity of the surface results in that we can capture its form and memorize it. With closed eyes we recall the figure at once in every detail. What is much more difficult is to verbally communicate the geometrical pattern to a third party. The adequate terminology is lacking.

心脏曲面

尽管心脏曲面方程$y^2+z^3=z^4+x^2z^2$非常简单，但心脏曲面却拥有一个微妙的局部和整体结构。它的奇异点的轨迹是一条直线，沿着这条直线，曲面自相交。原点0在这里是一个很有趣的点，心脏曲面与平面x=c的交线垂直于奇异点的直线。在此循环中，若c趋于0，则结果收缩至一个节点，一个漏斗就此形成。从远处看，我们看到一个圆形开口，与竖直的xy平面的交线实际上是一个圆。这个曲面非常简单使得我们能够抓住它的形态并且记住它。闭上眼睛，我们可以立刻记起它的每个细节。困难的是将他的几何图形用另外一种几何图形联系起来，找到合适的术语来描述它。

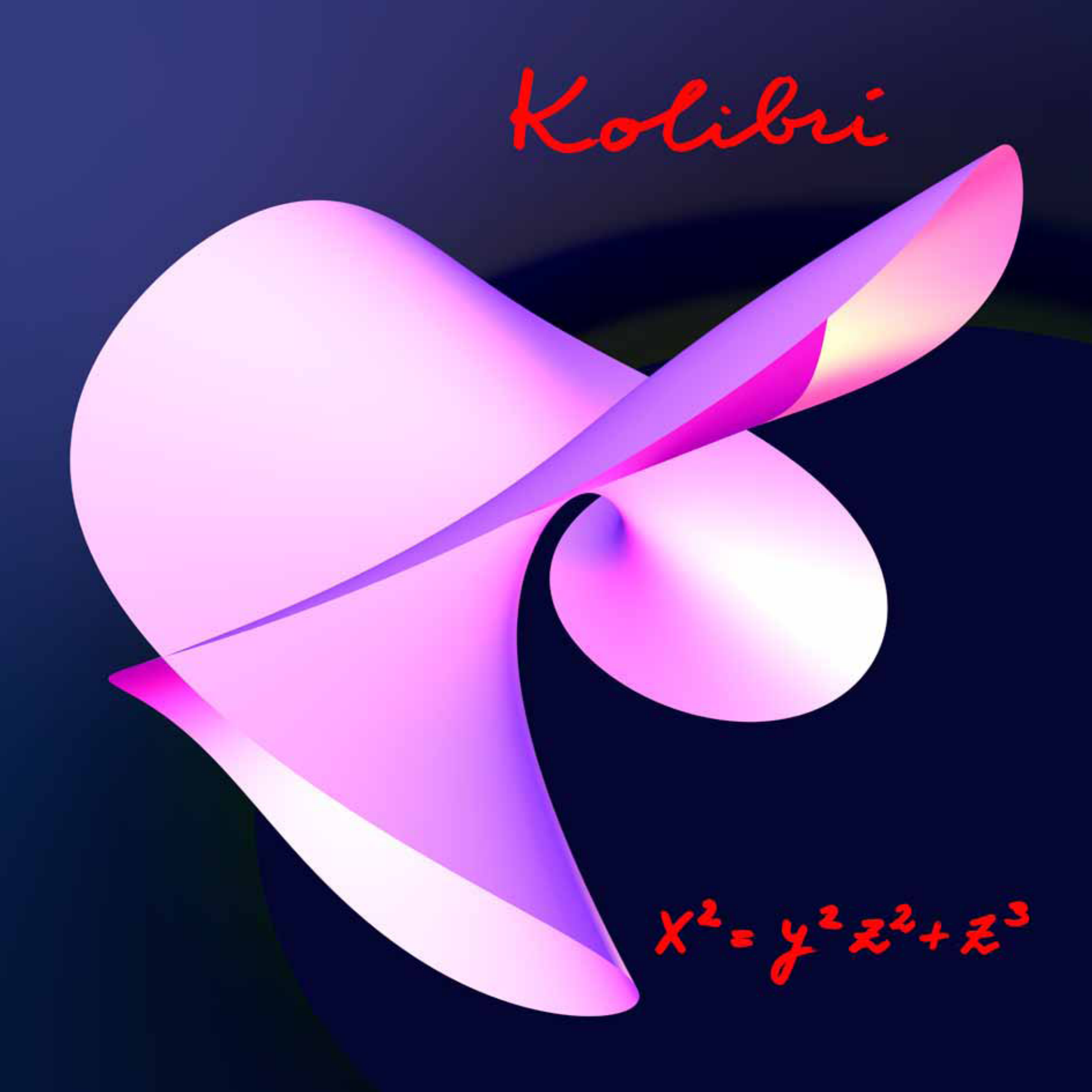
Herz
$x^2z^2+z^4=y^2+z^3$

HUMMINGBIRD

As you can easily see Hummingbird is a closeup view turned upside down of Heart near the origin. The equation is $x^3+x^2z^2 = y^2$, compare with the Heart formula $y^2+z^3 = z^4+x^2z^2$. If you substitute x by z and z by –x in the latter the monomial x^4 can be eliminated by a change of coordinates so that the Hummingbird equation comes out.
The Hummingbird as one of the smallest birds is equipped with impressing abilities. Its wings are able to beat up to 200 times in a second so that it can stand still in the air. This activity needs a lot of energy, that is why the hummingbird must eat twice its body weight in food per day. Without constant food supply it would starve within a few hours. It lowers its body temperature substantially at night so as to save energy.

蜂鸟

正如你很容易看到的那样，蜂鸟曲面是在原点附近倒置的心脏面的特写视图.考虑方程$x^3+x^2z^2=y^2$和心脏面方程$y^2+z^3=z^4+x^2z^2$相比较。如果在前一方程中将Z替换x，将-x替换z，然后通过坐标变换消去x^4，便可得到蜂鸟方程。蜂鸟随时最小的鸟类，却拥有非凡的能力。它的翅膀可以在1秒内扇动达到200次之多，因此它可以在空中不动。要做到这点需要大量的能量，这就是为什么蜂鸟要在一天中吃掉它两倍体重的食物。没有持续的食物补充，蜂鸟会在很短的时间内感到饥饿.蜂鸟也会在夜晚降低体温以保存能量。


Kolibri
$x^2 = y^2 z^2 + z^3$

SOLITUDE

The Solitude equation $x^2yz+xy^2+y^3+y^3z = x^2z^2$ does not reveal its hidden geometrical diversity. Similarly, the image shows only part of the phenomena. What is informative, however, is the camera drive round the surface as it can be experienced in the film "ZEROSET– I spy with my little eye".
There are obviously two openings, a larger well visible one and a smaller one, which you would not suspect from the first perspective. The view from above shows the vertical singular straight line alongside which horizontal sections force a sharp curve. The example of Solitude shows the complexity of the problem to deduce the visible real geometry from the equation. Similarly, one can question the underlying complex or number-theoretical geometry.

Solitude

Solitude方程$x^2yz+xy^2+y^3+y^3z=x^2z^2$没有透露其隐含的几何多样性。同样的，它的图形也仅仅显示了一部分此种现象.然而真正提供信息的是绕在表面的摄像头驱动，正如电影“零位调整-我用我的小眼睛侦察”中应用的那样。它有明显的两个开口，一个大的可以看到，一个小的不能够一下找到。从上而下的视图显示了一条竖直的奇异点的直线，沿着这条直线的水平截面是一条尖锐的曲线。这个例子说明了要从方程本身演绎可视的真实几何性征是非常复杂的。同样，任何人都可以质疑其中的复杂性和数值理论的几何形态

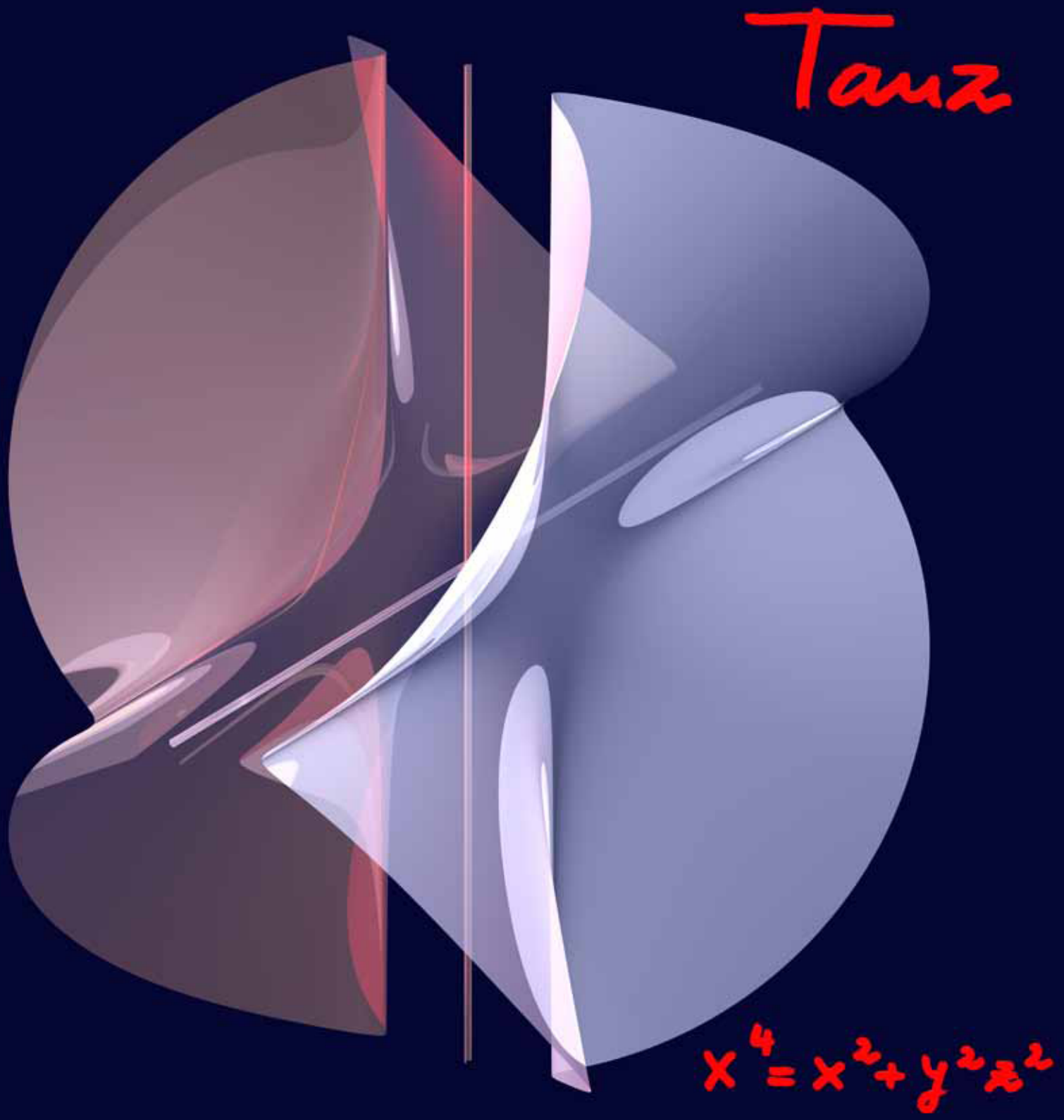
Solitude
$x^2yz + xy^2 + y^3$
$+ y^3z = x^2z^2$

DANCE

By setting $z = 0$ the Dance equation $2x^4 = x^2 + y^2z^2$ yields the equation $2x^4 = x^2$. This is the section of Dance with the horizontal xy plane. If you rewrite the equation you will get $x^2(\sqrt{2}x+1)(\sqrt{2}x-1) = 0$ of three parallel straight lines. We, here, observe a typically real phenomenon: The straight line $x = z = 0$ belongs to the solution set of the Dance equation $2x^4 = x^2+y^2z^2$, but is an isolated one-dimensional component, because if x is near 0 there are only the solutions $x = y = 0$ or $x = z = 0$, that means the cross consisting of the y and z axis.
As straight lines are infinitely thin they are missed by the visualization program or are not indicated. Therefore, the existence of one-dimensional components in the solution set must be cleared up by computation beforehand and then they must be added to the image as thin cylinders, if necessary.

舞蹈

令z=0，舞蹈方程$2x^4=x^2+y^2z^2$导出$2x^4=x^4$这是舞蹈方程与水平的xy-平面的截面，次方程可以改写为（√2x+1）（√2x-1）=0，即三条平行直线方程。这里，我们观察到一个真实的现象，直线x=z=0属于舞蹈方程的解集，但是是孤立的一维分支。因为当x接近0时，只有两组解x=y=0，x=z=0，就是y轴和z轴。因为直线可以无限细。因此，解集中的一维分支的存在性必须提前通过计划算确定，然后，有必要的话把他们用细线柱体的形式添加到图像中。


Tanz
$x^4 = x^2 + y^2 z^2$

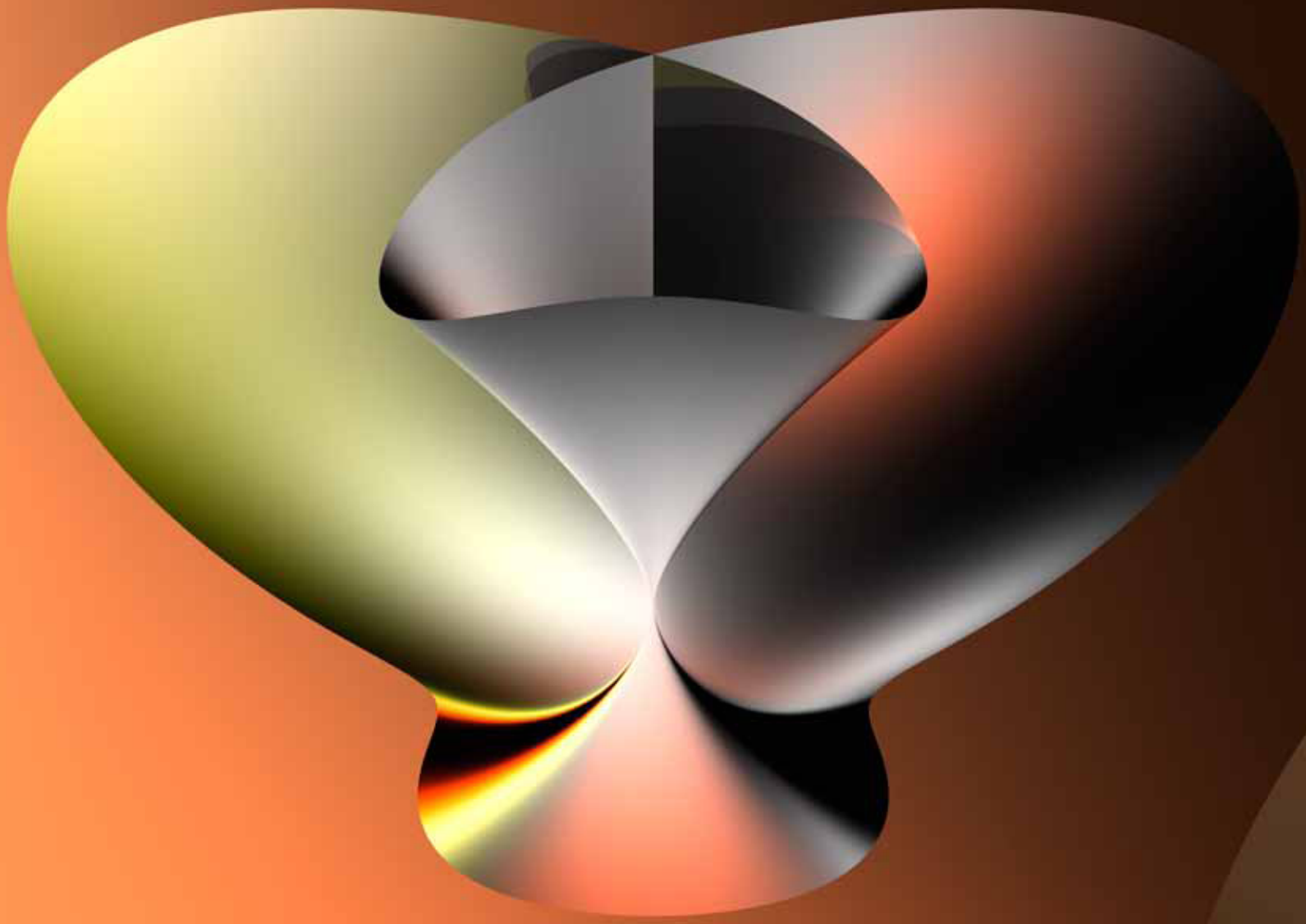
DOVE

Dove possesses the amazing formula $256z^3 - 128x^2z^2 + 16x^4z + 144xy^2z - 4x^3y^2 - 27y^4 = 0$. The coefficients are not incidental. On the contrary: The equation results from another more general formula, the so-called discriminant. It describes the shade of a surface or variety which emerges along with the projection on a surface or a linear space of higher dimension. The contour line is clearly defined by the surface and the projection and, from the algebraic point of view, also the form of the equation.

鸽子

鸽子拥有神奇的方程$256z^3-128x^2z^2+16x^4z+144zy^2z-4x^3y^2-27y^4=0$，它的系数并非偶然，相反，它的方程源于一个更一般的公式即所谓的判别式。他描述了曲面的阴影，或者是投影到一个曲面或者更高维的线性空间而出现的曲面簇。等高线可由曲面和投影精确定义，并且从代数角度，也可以给出方程的确切形式。

Taube



NOZZLE

The Nozzle surface is constituted by three smooth components, which intersect each other pairwise in a (likewise) smooth, plane curve. That way, three section curves are obtained, i.e. one straight line and two parabolas. Note that these curves touch each other tangentially at the origin. We here have the simplest example of a surface with three pairwise transversal components so that the section curves of every two of them do not intersect transversally. The surface, thus, is no *Mikado Variety*.
The transversal intersection of two smooth components of a surface – a fundamental concept of geometry - can, by way of the so-called ideal theory, be put into formulas in an algebraically precise manner. It can perfectly be used for calculation and proofs. In the case of singular components a correct definition of transversality is still to await.

Nozzle

Nozzle曲面有三个光滑分支组成，它们分别在光滑平面曲线上两两相交。 这样就得到了三条曲线，即一条直线和两条抛物线，注意到这些曲线在原点处相切，这里我们得到一个最简单的曲面实例：这个曲面有三个两两横截的分支，每两个分支的相交曲线都不横截相交。 因此，这个曲面不是Mikado簇。一个曲面的两个光滑分支横截相交（几何学的基本概念）可以利用所谓的理想理论用精确的代数公式表达。它可以完美的用于计算和证明。单一分支的情形还没有准确的横截定义。

Tülle
$yz(x^2+y-z)=0$

TICK

The simple Tick equation $x^2+y^2 = z^3(1-z)$ fully dictates the geometry just as with the other surfaces; that means, both the singular points and the outer shape, the curvature and the extension are clearly defined by the four monomials x^2, y^2, $-z^3$ und z^4. As a result, the formula is a very efficient way to codify forms which appear complicated. However, the geometric information cannot always be read from the formula. The local shape of the surface near a given point can be explicitly defined, in most cases; the techniques of local analytic geometry have a good effect. Defining the global structure requires much more efforts and cannot always be satisfactorily accomplished.

滴答曲面

就像其他曲面一样，滴答曲面的简单的方程$x^2+y^2=z^3(1-z)$充分表现了其几何特征； 也就是说不管是奇异点还是外形，抑亦是其曲率以及拓展都由四个单项式x^2，y^2 $-z^3$和z^4清晰的展示出来。 总之方程是表达看起来复杂形状的很有效的方式。但是几何信息也不总是都可以从方程中看出来。 大部分情形是曲面在给定点附近的形状会清晰的定义出来；也就是局部解析几何技术总是起到很好的表达效果。但是定义整体的结构即使花费更大的力气也未必能取得满意的表达效果。

Zeck
$x^2+y^2=z^2(1-z)$

NEPALI

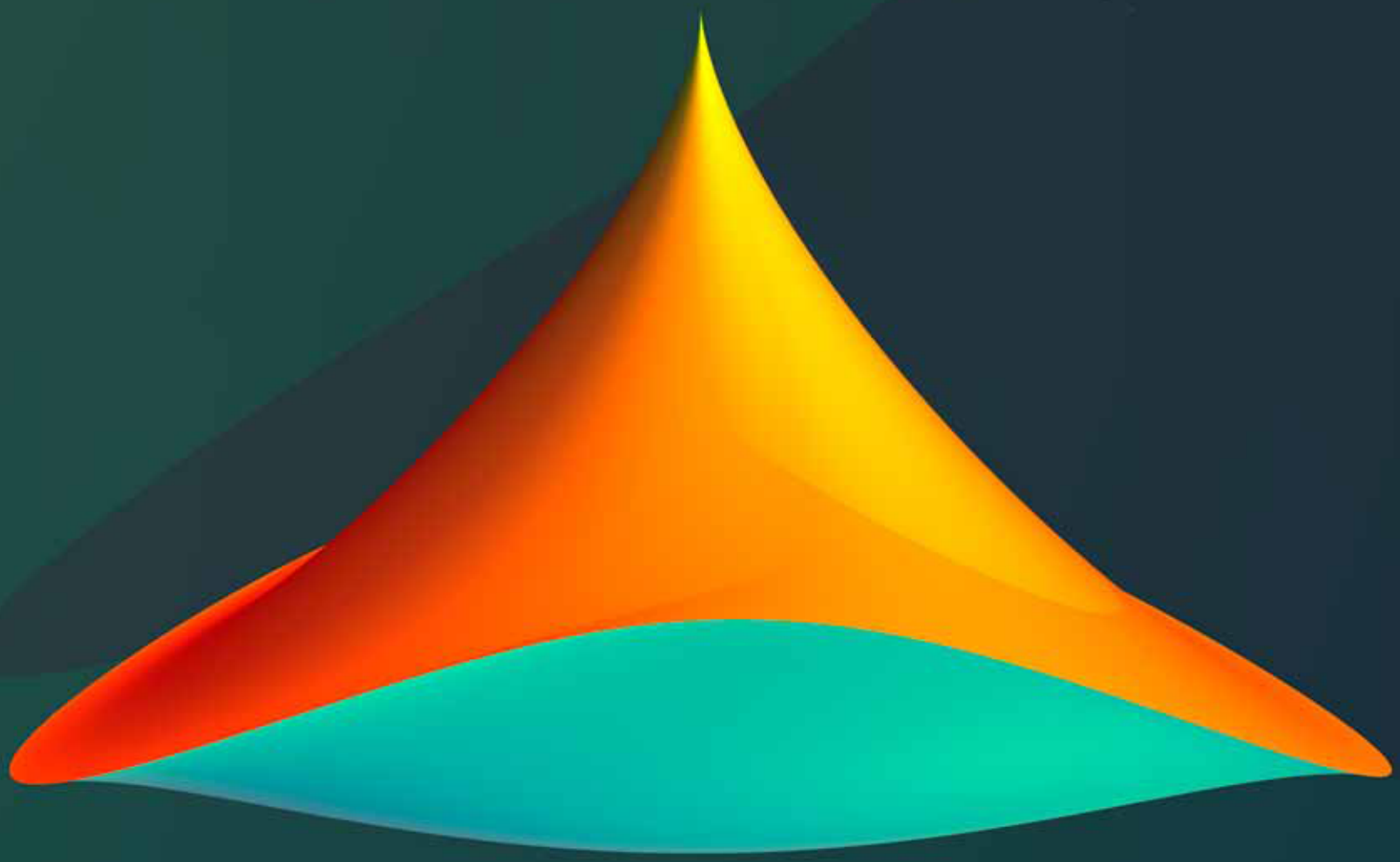
Let us look at the defining equation $(xy-z^3-1)^2 = (1-x^2-y^2)^3$ of Nepali. The symmetry between x and y is enforced by the quadratic polynomial x^2+y^2, which is rotation symmetrical in contrast to the monomial xy. Sections with horizontal planes $z = c$ yield closed curves being almost circles. The simultaneous occurrence of squares and third powers produce the tapering at the top. The lateral boundary curve of Nepali is no exact circle but is arching up and down like the brim of a hat. Its projection to the horizontal xy plane, however, is a circle as can be seen from the top view. The surface shown is bounded; hence there was no need of any cube or sphere intersection. This fact can be directly derived from the formula by accurate analysis.

尼泊尔曲面

让我们看看由Nepali定义的方程$(xy-z^3-1)^2=(1-x^2-y^2)^3$。二次多项式$x^2+y^2$规定了x与y之间的一个对称，与单项式xy规定的对称不同，这是旋转对称。Nepali曲面与水平面z=c的截断接近于圆。方程两边的平方和三次方产生曲面上方的尖锥形。Nepali曲面的外侧边界曲线不再是圆了，而是像帽檐一样向上和向下拱的形状。从上面看，它在水平坐标面面上的投影是个圆。所示的曲面是有界的，因此没必要给出所有的立方截断和球截断。这个事实可以由精确分析公式直接得到。

Nepali

$$(xy - z^3 - 1)^2 = (1 - x^2 - y^2)^3$$

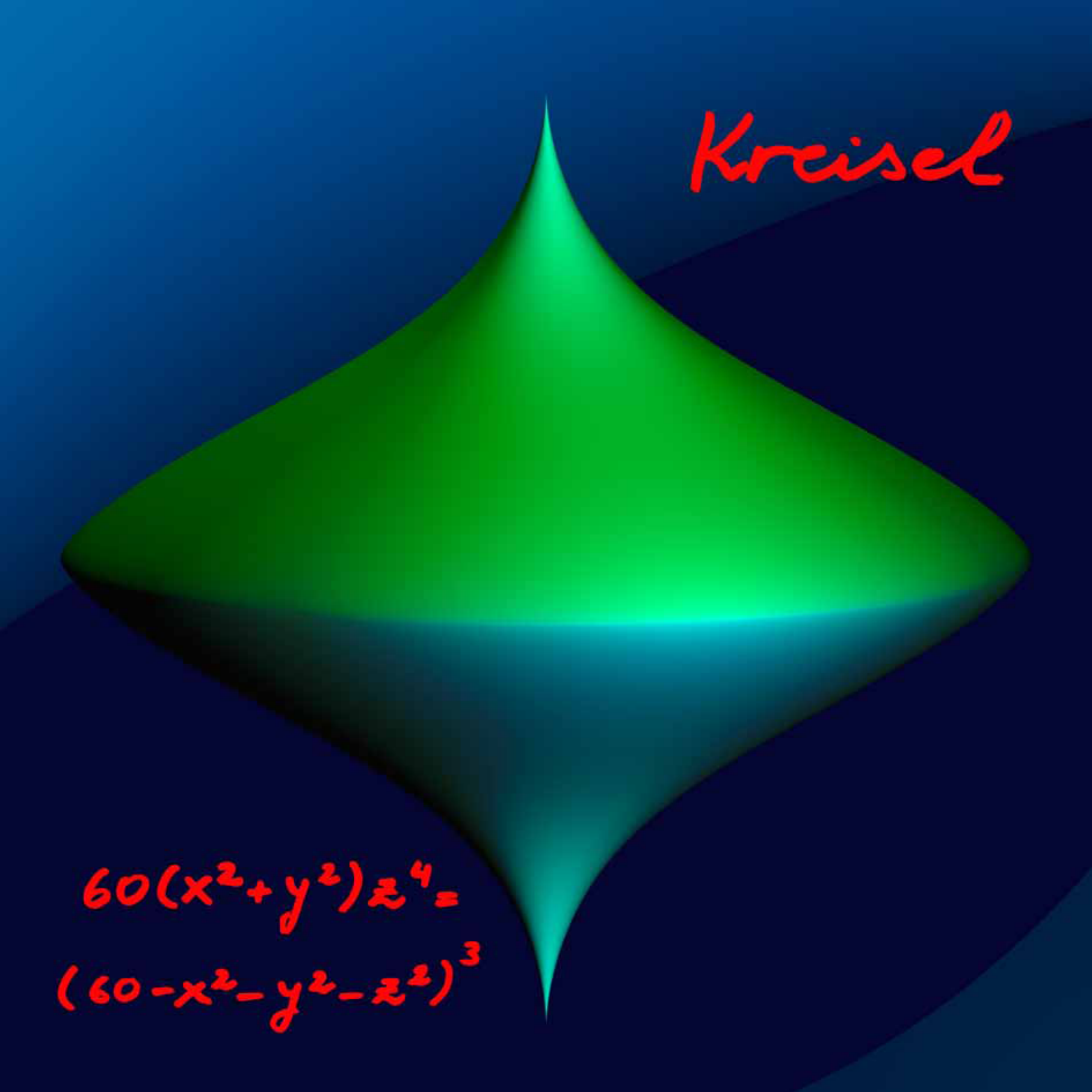


SPINNING TOP

To illustrate the shape of a spinning top as a solution set of a simple algebraic equation means to get kind of mathematical handicraft done. It is, indeed, not clear mathematically, at first sight, what this equation must look like irrespective of the fact that it does not need to be precise: Different equations can provide similar forms. The image shows the equation $60(x^2+y^2)z^4 = (60-x^2-y^2-z^2)^3$. The rotation symmetry round the vertical z axis is identified by x and y occurring in the quadratic polynomial x^2+y^2. The third power on the right hand side is necessary to produce the two apices. And as z occurs only in even powers the surface is symmetrical with respect to the reflection in the horizontal xy plane. The twofold parameter 60 to the balance of the proportions has aesthetic reasons.

陀螺面

为了展示某个简单代数方程的解集具有陀螺形就像做某件数学工艺品。因为不同的方程可以提供类似的曲面，所以如果不仔细考虑，数学上这个方程的曲面长相如何不能被一眼看出。以下是方程$60(x^2+y^2)z^4=(60-x^2-y^2-z^2)^3$的图示。变量x和y总是以二次多项式$x^2+y^2$的形式同时相伴出现的，因此曲面关于垂直的z轴旋转对称。方程右边的三次方给出曲面的两个尖端。变量z总是以偶次方的形式出现，因此曲面关于xy面对称。为了美观，用二重参数60平衡了一下比例。


Kreisel
$60(x^2+y^2)z^4 =$
$(60-x^2-y^2-z^2)^3$

DIABOLO

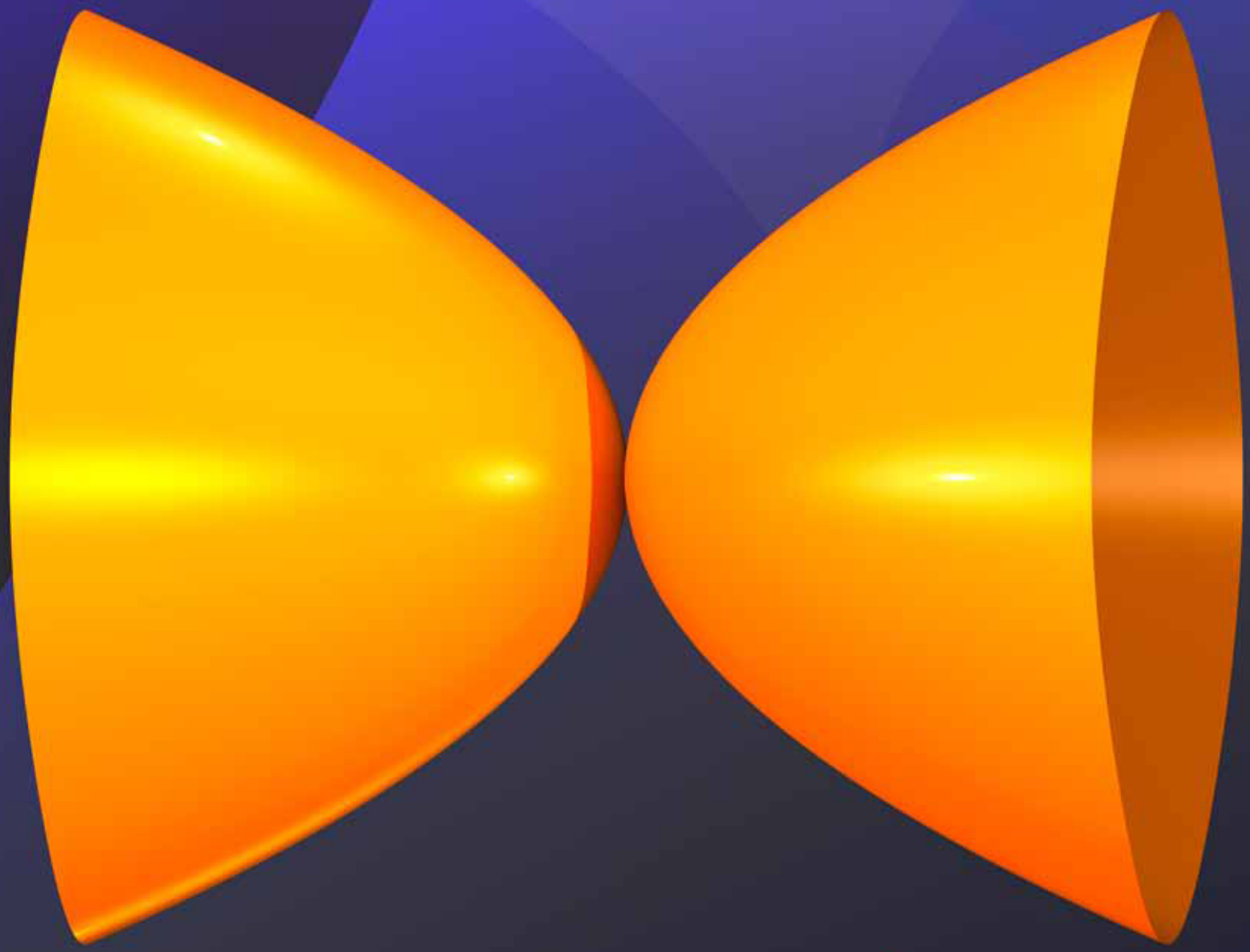
The Diabolo equation $x^2 = (y^2+z^2)^2$ factorizes into the product $(x-y^2-z^2)(x+y^2+z^2) = 0$. Hence, the respective surface is the unification of the two single-shell rotating hyperboloids $x = \pm(y^2+z^2)$. They touch each other tangentially at the origin. The contact can algebraically be gathered from the concurrent linear term x in the two factors. The tangent plane is the vertical plane $x = 0$.
The stripes in the images are shades due to lighting. If you modify the Diabolo equation by adding a constant term such as in $x^2 = (y^2+z^2)^2+1/1000$, then the two halves are separated. The substitution of x by x+y, however, yields the variation $(x+y)^2 = (y^2+z^2)^2$ of the equation. The two shells are moved at an angle to each other.

空中陀螺

空中陀螺曲面方程$x^2=(y^2+z^2)^2$ 可以被因式分解成$(x-y^2-z^2)(x+y^2+z^2)=0$。所以它对应的曲面$x=\pm(y^2+z^2)$是两个单叶旋转双曲面 的并。这两页曲面在原点相切。切点可以由两个因子联立方程组，然后消x元计算得到。切平面是铅直坐标面x=0 。

插图中亮颜色的条纹用来表现光。稍稍修改一下Diabolo方程，比如加个常数后得$x^2=(y^2+z^2)+1/1000$，这时两叶曲面就分开了。用x+y代换x后可以得到方程的另一个变形$(x+y)^2=(y^2+z^2)^2$，此时两页曲面按一个角度相互错开。

Diabolo



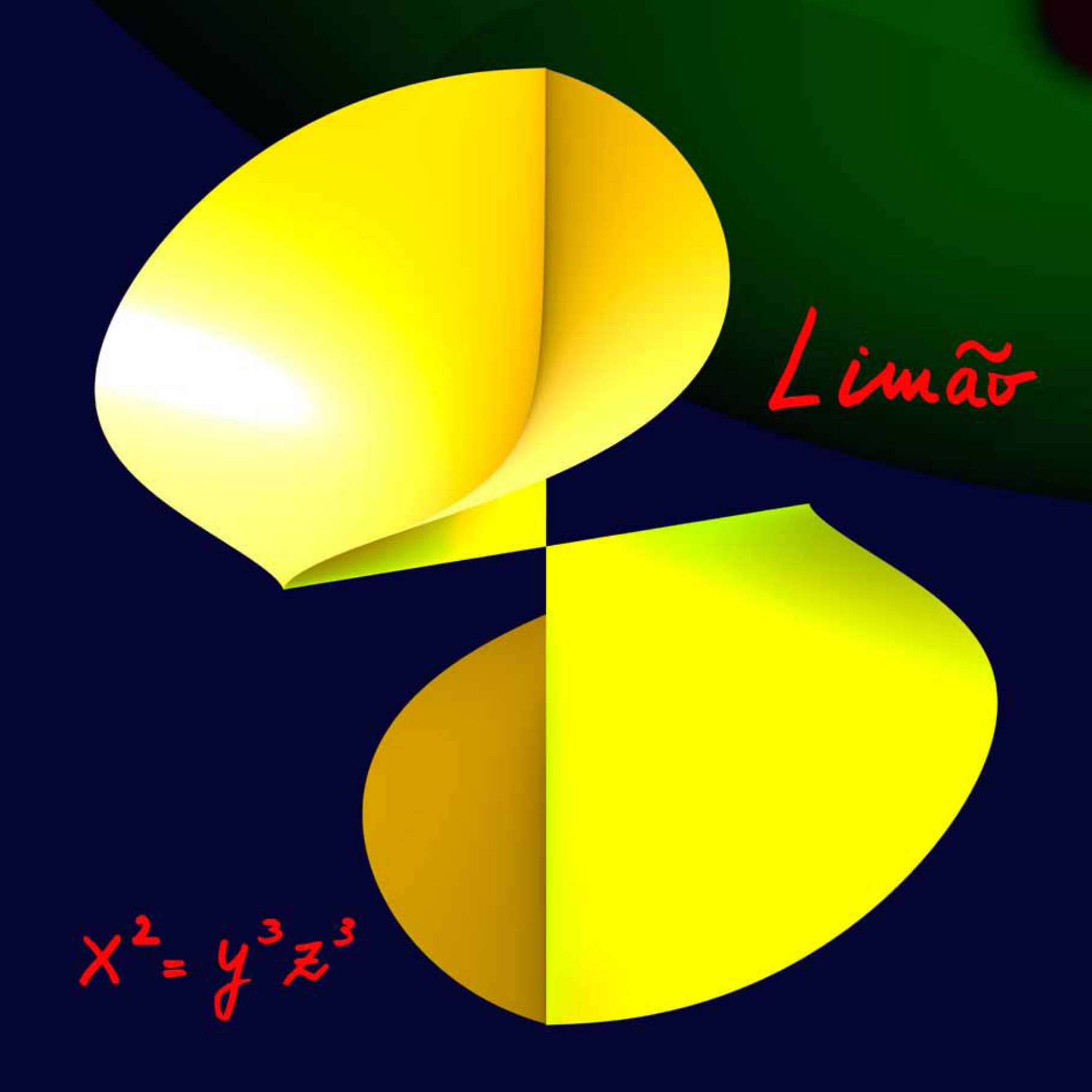
$$x^2 = (y^2 + z^2)^2$$

LIMÃO

This surface with the equation $x^2 = y^3z^3$ is difficult to visualize in an appealing manner. Both the lighting and the shaping near the origin produce problems, since the image shall present the figure as authentically as possible. This depends on the geometry and above all on the viewing angle of the camera. Artifacts occur, now and again, such as frayings or mottled colourings, which do not correspond to mathematical reality. Also, reflections can have a disturbing effect. Alongside the edges, mist is coming up near the intercept point, despite the quality of the POVRay program. The problem rather lies in the complexity to solve equations near singular points. The so-called resolution of singularities, which provides a parametrization of the surface, can help in many cases.

立马奥曲面

把方程$x^2=y^3z^3$的曲面可视化有些困难，这也使得这项工作十分有吸引力。因为图像要尽可能真实的体现数学对象，原点附近的光和形就成了问题。成像取决于几何，尤其是摄像头的观看角度。艺术加工，比如上色和打磨，时常不依赖于数学事实。而且反射可以制造麻烦。尽管POVRay程序的质量很好，截距点和边缘还是有些模糊。作图的困难归根结底还是来自于奇点附近解方程的复杂性。奇点所谓的“解”，也就是曲面的一个参数化，在很多方面都有用。

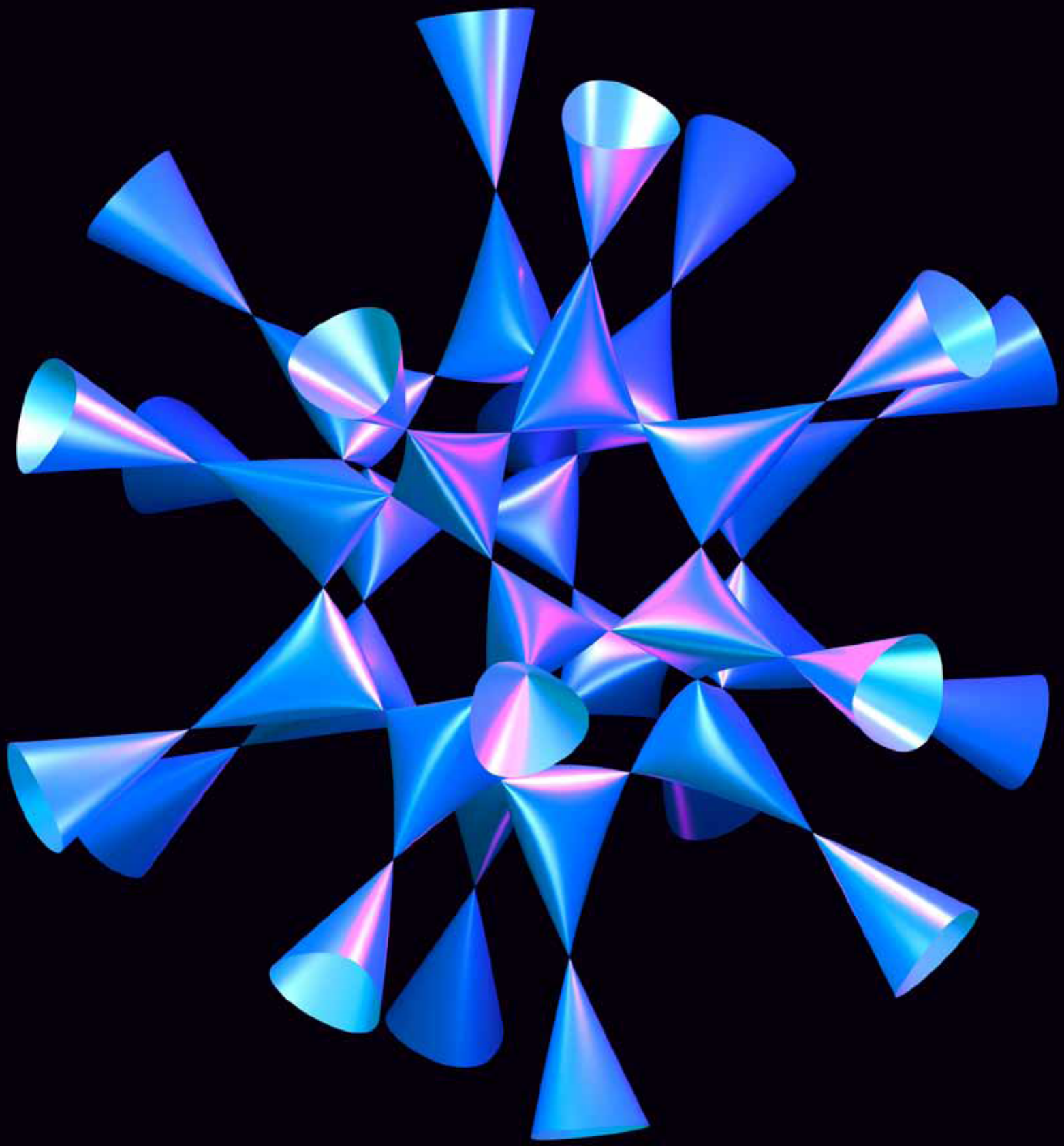

Limão
$x^2 = y^3 z^3$

THE BARTH SEXTIC

This surface of degree 6 (sextic) was constructed by Wolf Barth in 1996. Altogether, it has 65 singularities when also counting the 15 invisible ones which are infinitely far away. 65 is the maximum possible number of singularities on a sextic as shown in 1997 by Jaffe and Ruberman. Barth's construction was a big surprise: For a long time geometers believed that a surface of degree 6 cannot have more than 64 singularities. The icosahedral symmetry is one of the most striking features of Barth's sextic. But not all sextics with 65 have this kind of symmetry; there even exists a 3–parameter family of surfaces with 65 singularities! In this family one can choose three parameters almost at random and always obtains a sextic with 65 singularities. The exact equation of Barth's sextic is $P_6 - \alpha K^2 = 0$, where $P_6 = (\tau^2x^2-y^2)(\tau^2y^2-z^2)(\tau^2z^2-x^2)$, $\tau = 1/2(1+\sqrt{5})$ is the golden ratio, $\alpha = 1/4(2\tau+1)=1/4(2+\sqrt{5})$ and $K = x^2+y^2+z^2-1$ describes a sphere of radius 1.

巴斯六次曲面

这个6阶曲面是由沃尔夫巴斯在1996年构造的。它总共有65个奇点，其中看不见的15个位于无穷远处。1997年 Jaffe 和Ruberman证明了6阶曲面至多有65个奇点。很久以来几何学家们相信6阶曲面的奇点数不会超过64，因而巴斯的构造令人大吃一惊。巴斯6阶曲面的最显著特征之一是具有正20面体对称。但是不是所有具有65个奇点的6阶曲面拥有这种对称性。实际上存在一个带3个参数有65个奇点的曲面族。这族曲面中，我们几乎可以任意取定3个参数而后得到一个65个奇点曲面。巴斯给出的方程是P_6-ak^2=0，其中$P_6 = (\tau^2x^2-y^2)(\tau^2y^2-z^2)(\tau^2z^2-x^2)$, $\tau = 1/2(1+\sqrt{5})$，是黄金分割率，$\alpha = 1/4(2\tau+1)=1/4(2+\sqrt{5})$ and $K = x^2+y^2+z^2-1$，是半径为1的球。

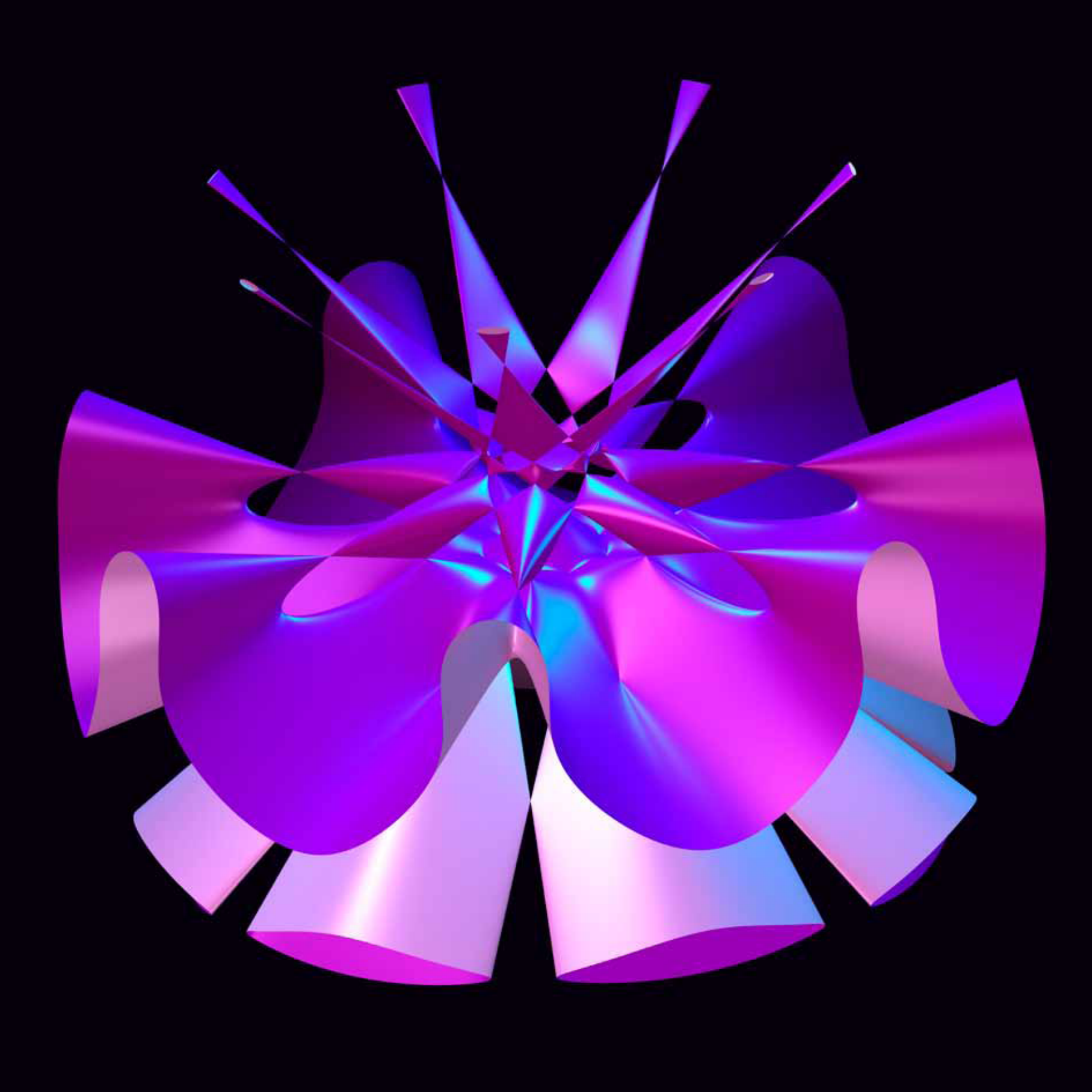


THE LABS SEPTIC

In 2004, while writing his Ph.D. thesis at Mainz, Oliver Labs constructed a surface of degree 7 (septic) with 99 singularities. Since 1997 one knows for surfaces of degree 6 that the maximum possible number of singularities is 65. In degree 7 there is a result of A.N. Varchenko from 1982 which tells us that there cannot be more than 104 singularities. In 1992, Chmutov constructed a septic with 93 singularities which was the world record at that time. Since Labs' construction, this world record has been 99. It is open if septics can have 100, 101, . . . , 104 singularities.
The Labs Septic has the symmetry of a regular 7–gon. However, similar to Duco van Straten's computation for the sextics, one can compute that there is indeed a 5–parameter family of septics with 99 singularities. For the construction of the septic Oliver Labs used the computer algebra software which is developed at the technical university of Kaiserslautern. This software is very well suited for working with singular algebraic surfaces.

莱布斯七次曲面

2004年，奥利弗莱布斯在美因兹（Mainz）写他的博士论文时，构造了一个带99个奇点的7阶曲面。自从1997年以来，人们已经知道6阶曲面至多有65个奇点。关于7阶的，A.N. Varchenko 在1982年给出一个结果中说不会超过104个奇点。1992年 Chmutov 构造了一个带93个奇点的7阶曲面，这是当时的世界记录。莱布斯的构造将世界纪录提升至99。至于是否存在有100、101、...、104个奇点的7阶曲面，这个问题仍然是开放的。莱布斯7阶曲面具有正7边形对称。然而，类似于Duco van Straten 对于6阶曲面的计算，我们可以算出存在一个带5个参数的有99个奇点的7阶曲面族。奥利弗莱布斯在构造7阶曲面是使用了代数软件SINGULAR,它是由凯泽斯劳腾理工大学（Technical University of Kaiserslautern）开发的。这个软件对于研究奇异代数曲面十分适用。

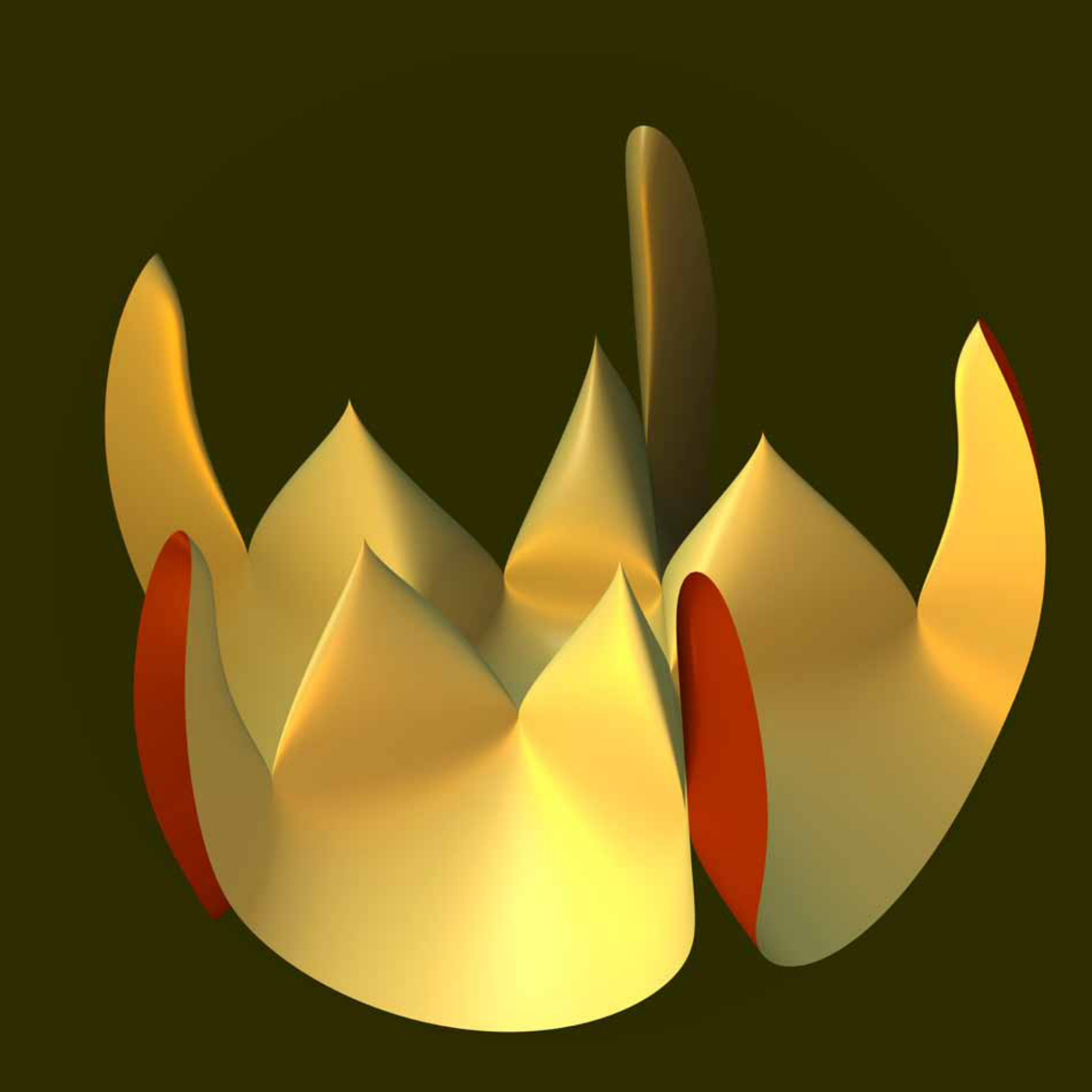


A QUINTIC WITH 15 CUSPS

This surface of degree 5 (quintic) has 15 singularities whose type is called ordinary cusp or A2. The surface is part of a series of related surfaces of infinitely many degrees constructed by Oliver Labs in 2005. As one can see from the picture, five of the singularities look different than the others. These five ones are more specifically of type A_2^{++} and the others of type A_2^{+-}. The former can be described locally by the equation $x^3+y^2+z^2 = 0$, the others by $x^3+y^2-z^2 = 0$. The equation of the quintic with 15 cusps is $S_5(x, y)+t(z) = 0$, where $S_5(x, y) = x^5-10x^3y^2+5xy^4-5x^4-10x^2y^2-5y^4+20x^2+20y^2-16$ is a regular pentagon, and where the polynomial $t(z) = -3z^5+10z^3-15z-8$ is a variant of the so–called Tchebychev Polynomials.

带15个尖点的5次曲面

此5阶曲面拥有15个称之为常规尖点或A2型的奇点。奥利弗莱布斯于2005年构造了一系列相关的无穷多阶曲面。这个曲面是其中之一。在图像中我们可以看到，有5个奇点与其它不同。这5个特殊奇点是A_2^{++}型的，其余的是A_2^{+-}型的。前者可以由方程$x^3+y^2+z^2=0$的局部描述，后者则由$x^3+y^2-z^2=0$描述。图像曲面的方程是$s_5(x,y)+\tau(z)=0$，其中$s_5(x,y)=x^5-10x^3y^2+5xy^4-5x^4-10x^2y^2-5y^4+20x^2+20y^2-16$是一个正5边形，而多项式$t(z)=-3z^5+10z^3-15z-8$是单变量车比雪夫多项式。



LUC BENARD

Luc Benard is a Canadian who spent his whole life in Montreal. He worked as a cinetechnician, cameraman, and audio engineer and is currently doing electronic editing. Due to his passion for science and art he started to look more closely at it focusing on plasma physics, high current, electronics and information technology. With computers becoming more and more advanced he began to use fractals as a basis to his visual work. In the last few years he has been working on 3D images and has tried to combine science and art in his visual compositions.

卢克伯纳德

卢克伯纳德，加拿大人，一直都生活在蒙特利尔。他做过电影技术员、摄影师、音频工程师，现在做电子剪辑。 出于对科学和艺术的热爱，他开始专注于研究等离子物理学、高电流、电子和信息技术。随着计算机技术越来越先进，他开始以分形为基础做数学的可视化工作。在过去的几年中他一直致力于 3D 图像的制作。在他的视觉作品里，卢克试图将科学与艺术结合在一起。

RICHARD PALAIS

Richard Palais, Professor Emeritus at Brandeis University, is a well-known mathematician who retired early to work on mathematical visualization. He is the leader of an international team of mathematicians, the 3DXM Consortium, and the chief architect and programmer of their software, 3D-XplorMath (http://3D-XplorMath.org). He is currently in the math department at the University of California at Irvine, and together with Luc Benard and other consortium members he is working on creating and maintaining a “Virtual Math Museum”.

理查德 帕莱斯

理查德 帕莱斯，布拉迪斯大学的退休教授，知名数学家。为了从事数学可视化工作他提前退休。他正领导着一个由国际数学家志愿组成一个团体：3DXM。该小组研发了软件3D - XplorMath（http://3D-XplorMath.org）。他还是该团体的首席设计师和程序员。如今他任职于回话大学分校的数学系。与卢克伯纳德以及团体的其他人员共同创建了并维护着“虚拟数学博物馆”

THE VIRTUAL MATH MUSEUM

Mathematicians have used advances in the graphic arts to improve how they display mathematical concepts in their teaching and research going back to that ancient "upgrade" from drawing diagrams in sand to incising them in clay tablets. So it is no surprise that the paradigm changing modern technology of computers has provided spectacular new possibilities for visualizing ever more complex mathematical objects and processes. 3D-XplorMath is a math visualization program designed to exploit fully this new technology. It is in effect a virtual math museum with an intuitive user interface that allows even non-mathematicians to take advantage of the new graphic possibilities to experience the visual beauty inherent in many different fields of mathematics.

It came as a welcome surprise to its developers that a number of artists were sufficiently intrigued by the aesthetic possibilities they detected in many of the images produced by 3D-XplorMath that they incorporated them as the subject matter in a new genre of artistic imagery. Luc Benard is an outstanding member of this new class of mathematically oriented artists. While Luc specializes in exposing the beauties of fractals and surfaces, he feels that every area of mathematics has its own beauty, to be found in the elegance of its concepts, theories, and equations, and in the wonderful imagery associate with them. Several of his creations are included in this exhibition, and Richard Palais recruited him as a member of the 3DXM Consortium, asking him to create a gallery of mathematical art for its Virtual Math Museum.

虚拟数学博物馆

从古代开始，数学家们就在教学和科研中利用图像的便利来展示数学概念。他们在沙上画图，用黏土雕塑。今天更先进的计算机技术提供了展示更复杂数学对象和数学过程的可能性。其结果也更为壮观！充分利用新的计算机技术，3D - XplorMath软件是一个数学可视化程序。实际上，它是一个带有直观拥护界面的虚拟数学博物馆。允许非数学专业的人通过新的图像来体验不同数学领域的内在美。

软件的研发者们很吃惊于许多艺术家都痴迷于这项技术。艺术家们利用3D - XplorMath创制了很多图画，借此他们探索其中的美学潜力，成为一个新的艺术流派。Luc Benard是这个年轻的以数学为目标的艺术流派中的佼佼者。虽然Luc专攻分形和曲面，他感觉到数学的其它领域都有其内在美。数学概念、理论和方程中的优雅还有待发现。展览展示了他的几个作品。Richard Palais已经吸收他为3DXM团体的成员，并让他为虚拟数学博物馆创做一个数学艺术画廊。

LINKS
http://3D-XplorMath.org
http://virtualmathmuseum.org
http://virtualmathmuseum.org/mathart/MathematicalArt.html

STILL LIFE: FIVE GLASS SURFACES ON A TABLETOP

This image is the result of a collaboration between mathematician Richard Palais and graphic artist Luc Benard. This image was awarded first place in the illustration category of the National Science Foundation/Science Magazine 2006 Visualization Challenge and was on the cover of the 22 September 2006 issue of Science. The five mathematical surfaces depicted are (starting from lower-left and moving clockwise): the Klein Bottle, the Symmetric 4-Noid, the Breather Surface, the Boy Surface and the Sievert-Enneper Surface. The surfaces in the image were created using the 3D-XplorMath program developed by Palais, and assembled and rendered in Bryce by Luc Benard.

静物：桌面上的5类曲面

下面的图是数学家Richard Palais 和图像艺术家Luc Benard合作的结果。此图像荣获全国科学基金会 / 《科学》杂志 2006年可视化图类挑战赛中的第一名，并被刊登在 2006 年 9 月 22 日的《科学》杂志封面上。描绘的五个数学曲面分别是（从左下角开始，按顺时针方向）：克莱因瓶、 对称4-Noid，Breather曲面、 Boy曲面、Sievert-Enneper 曲面。图中的曲面是由 3D-XplorMath 软件绘制的。此软件由Palais 研发并由 Luc Benard 在 Bryce 组装和呈现。

LYAPUNOV PLAY

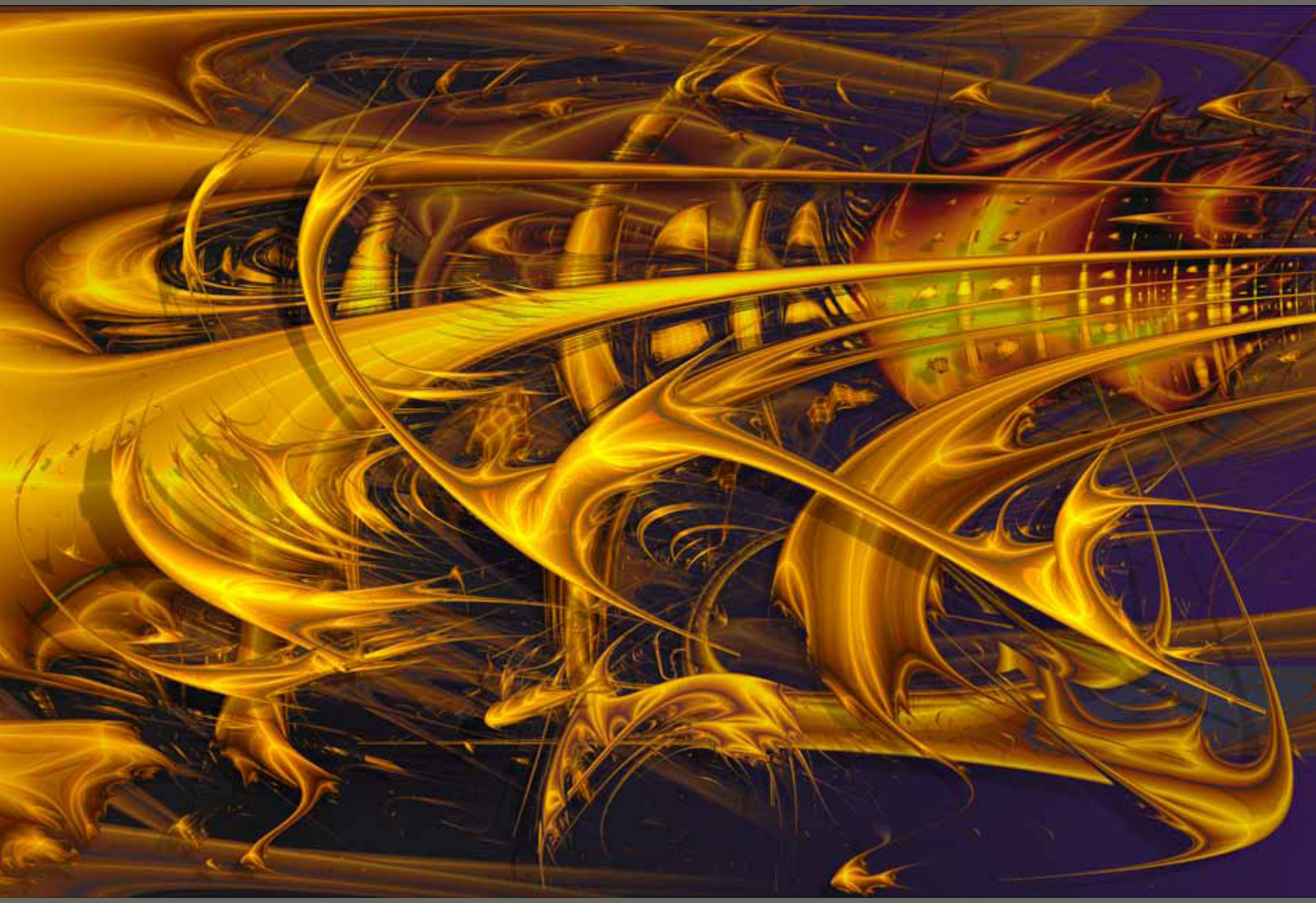
Mario Markus of Max Planck Institute for Nutrition has used dynamical systems to study the evolution of animal populations - the change over time of food, fertility, size, etc. - with dynamics requiring the ability of reproduction to alternate quasi-periodically between two values. Such systems can show both a stable cycle and chaotic evolution depending on the fertility ability. Stability or chaos can be analysed by computing the so-called Lyapunov exponent. (Lyapunov was a Russian mathematician living at the end of the 19th century.)

Markus-Lyapunov images are colour mappings of the Lyapunov exponent versus fertility, along horizontal and vertical axes. Only the stability domain is plotted; here, chaos (i.e. positive Lyapunov exponent) is rendered in dark blue. As the exponent goes from 0 to minus infinity, shades range from light to dark. At zero, the chaos threshold, the colour suddenly jumps from dark blue to a lighter shade. There is clearly much that is arbitrary in this colour mapping, and this gives an opportunity for choices based on aesthetic considerations. The picture consists of seven original Markus-Lyapunov pictures which were rebuilt and superimposed.

李雅普诺夫剧

马里奥 · 马库斯的马克斯 · 普朗克营养协会利用动力系统来研究了动物种群的演变，如一段时间内关于食物、 生育率、 体形大小等的变化。这种动力学需要繁殖能力能在两个值之间拟周期变化。这种系统可以显示依赖于生育能力的稳定周期和混沌演化。稳定或混沌可以通过计算所谓的李雅普诺夫指数来进行分析。(李雅普诺夫是19 世纪末的俄罗斯数学家)。

马库斯-李雅普诺夫图像是李雅普诺夫指数对于生育率沿水平和垂直轴的颜色映射。在这里，只有稳定域的绘制，深蓝色呈现混乱 (即李雅普诺夫指数为正时)。随着指数从 0 变到负无穷大，底纹范围由亮变到暗。在零点时，即混沌产生之际，颜色突然由深蓝色变为一个更亮的底纹。很明显，这在颜色映射中是任意的，这样就有机会基于审美考虑而作出选择。图片由七个原始的 马克思 李雅普诺夫图片重新生成并叠加而成。



EQUATIONS STUDIES

These fractal images were produced by using various equations as input in Stephen Ferguson's Flarium24 Windows program, using 40 iterations and the filter

*rr+=atan(fabs(dzy/dzx))*atan(fabs(dzx/dzy))*2.*

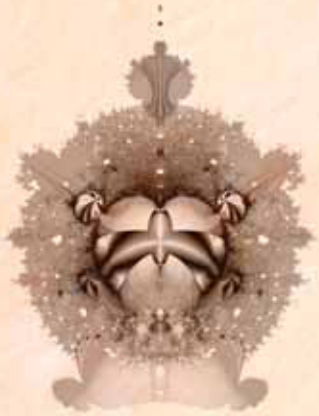
They were then assembled in Photoshop using fonts and coloration to mimic the old technical drawings of Leonardo DaVinci. While these images are highly mathematical in origin, many peoples find them aesthetically appealing completely independent of their source.

方程研究

在Photoshop中收集这些图片利用文字和着色模仿达芬奇的技术绘图。这些分形图像是通过在斯蒂芬 · 弗格森的 Flarium24 Windows 程序中输入各种方程经过40次迭代和过滤。

rr + = atan(fabs(dzy/dzx))*atan(fabs(dzx/dzy)) * 2

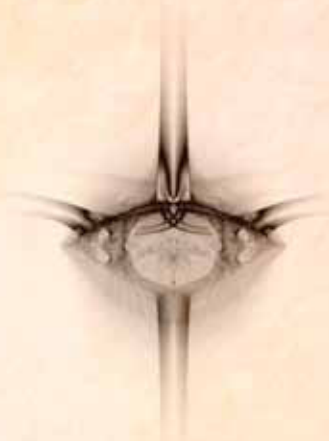
虽然这些图像本质上是高度数学化的，但很多人发现了它们从美学上也是非常具有吸引力的，与其来源没有任何关系。



$z = ((z^3 - z - 1)/$
$(z^3 - 1) - z) \times c$

$z = z - (z^4 - z)/$
$(4 \times z^3 - z)$
$z = z \times c$

$z = ((1 - z - z^2 \times z)/$
$(z - (4 \times z^2)) - z) \times c$
$z^2 = z^3$



$z = z - (z^2 - z)/$
$((3 \times z) - 1) + c$

$z = z^2/c + z$

$z = z \times c + c^6$

$z = z \times z + c$

WADA BASINS

This is a rendering that replicates the results of experiments in chaotic scattering. The basic setup is four identical, highly reflective balls sitting in a pyramid formation so that each ball touches every other ball. If you look into the gaps between three balls, the reflected images you see make up a three dimensional fractal that has what is known as the Wada Property after a Japanese mathematician who studied such spatial divisions in 1917. The Wada Property refers to a discrete dynamical system having three basins of attractions that are so intertwined that every point on one basin boundary is also on the boundary of all other basins. The image was assembled and rendered in Bryce.

Wada Basins

这里的图片呈现的是类似于混沌散射实验中产生的结果。它是由四个完全一样的反光球以金字塔的形式放置而成的，其中每个球都与另外三个球相互接触。如果你看三个球之间的空隙，你所看到的反射图像就组成一个三维的分形。该分形具有Wada 性质，其中 Wada是日本数学家，他曾在1917年研究了这种空间分割。Wada性质指的是一种具有三个吸引盆地的离散动力系统，这些吸引盆地相互交织且每个吸引盆地边界上的每一点也在其他两个吸引盆地的边界上。这个图像是在Bryce中组装和呈现。

A TRIPLY-PERIODIC LEVEL SURFACE

The green-tinted surface pictured here has three orthogonal translational symmetries. It is the level-surface given by the trigonometric equation:

$$4*(\cos x*\cos y+\cos y*\cos z+\cos z*\cos x) - 3*\cos x*\cos y*\cos z = -2.4$$

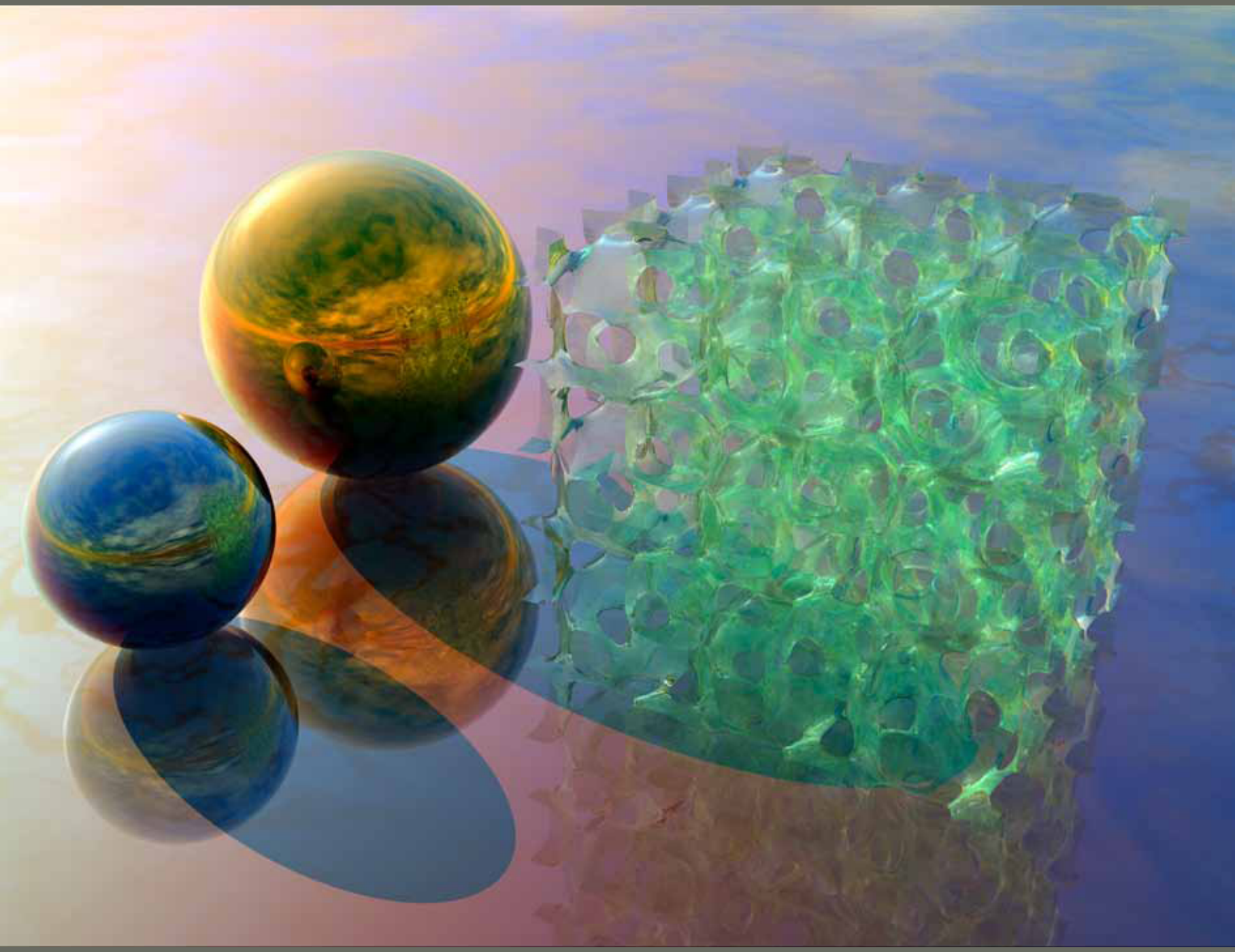
A unit cell may be viewed as a central chamber with tubes to the corners and faces of the cube. This surface closely approximates the minimal P-Surface discovered in 1880 by Karl Hermann Amandus Schwartz (who was a Professor at Halle, Göttingen and Berlin). Recently it has been investigated by material scientists, who use it and related surfaces to model so-called block-copolymers. The original model was taken from David A. Hoffman. This was textured, placed in a scene that we feel adds to the beauty of the object and rendered in Bryce.

三周期水平曲面

这里表面绿色的曲面具有三个正交平移对称。它是下面三角方程所确定的水平曲面：

$$4 * (\cos x * \cos y + \cos y * \cos z + \cos z * \cos x) -3 * \cos x * \cos y * \cos z = -2.4$$

每个单元格可被视为带有管角和立方面的中央房。这个曲面很接近由卡尔 · 赫尔曼 · 阿曼德 施瓦茨(哈雷、 哥廷根和柏林的教授)于1880 年发现的极小P表面。最近有材料科学家研究并利用它和相关的曲面建立所谓的嵌段共聚物模型。原始模型来自 David A.霍夫曼。我们认为应该在这美丽的对象上添加纹理，在Bryce中加以呈现。



Charles Gunn

Charles Gunn studied mathematics and computer graphics at the University of North Carolina where he was awarded Master of Science, in 1983. He was employed with Tektronix, Lucusfilm and Pixar and then worked with the Geometry Center of the University of Minnesota for five years. Since 2003 he has worked in the working group of Mathematical Visualization at TU Berlin. Apart from the development of software for mathematical visualization and animation, his main scientific interest is on projective and non-Euclidean geometry.

Charles Gunn

Charles Gunn在北卡罗来纳大学研究数学和计算机图形学，在1983年被授予理学硕士。他受聘于Tektronix、 Lucusfilm 和Pixar，然后在明尼苏达大学的几何中心工作了五年。自 2003 年以来，他一直在德国柏林工业大学的数学可视化工作组中工作。除了数学可视化和动画软件的开发，他主要的科学兴趣是在射影几何和非欧几何。

Tim Hoffmann

Tim Hoffmann studied mathematics at TU Berlin. After his degree in 1995 he did academic work in Berlin, Amherst / USA, Jena and Munich. He did his doctor's degree under Ulrich Pinkall in 2000 at TU Berlin, and in 2003 qualified as a university professor, also in Berlin. He has worked on discrete differential geometry, mathematical visualization and the development of mathematical software since the mid 90ies. He assisted in initiating the visualization system jReality.

Tim Hoffmann

Tim Hoffmann在德国柏林工业大学研究数学。1995 年拿到学位后，他在柏林，阿默斯特 / 美国，耶拿和慕尼黑做学术研究工作。2000年他在德国柏林工业大学Ulrich Pinkall教授的指导下完成了博士学位，并于 2003 年在柏林被授予大学教授资格。自90年代中期他一直从事离散微分几何、 数学可视化和数学软件的开发。他协助开发了可视化系统 jReality。

Ulrich Pinkall

Ulrich Pinkall studied mathematics at Freiburg University and did his doctor's degree in 1982. Since 1986 he has been professor in differential geometry and visualization at TU Berlin, he was spokesman for the Collaborative Research Centre "Differential Geometry and Quantum Physics" from 1992 to 2003. Together with John Sullivan, he has headed the working group "Mathematical Visualization" at TU Berlin since 2004, which is also part of the DFG Research Centre Matheon. His main field of activity is differential geometry of surfaces, also using virtual reality technologies.

Ulrich Pinkall

Ulrich Pinkall在弗赖堡大学研究数学，并在 1982 年取得博士学位。从1986年开始，他一直是德国柏林工业大学微分几何和可视化方向的教授，从1992 年至 2003 年，他还是协作研究中心"微分几何和量子物理学"的发言人。2004 年以来，他与John Sullivan领导德国柏林工业大学的"数学可视化"工作组，这也是德意志研究中心 Matheon 的一部分。他的主要研究领域是曲面的微分几何和虚拟现实技术。

Nicholas Schmitt

Nicholas Schmitt studies the differential geometry of constant mean curvature surfaces, their construction and classification.
Much of his research involves the development of software for investigating such surfaces experimentally. Schmitt obtained his doctorate from the University of Massachusetts, Amherst, in 1993. He has continued his research at the Center for Geometry, Analysis, Numerics and Graphics in Amherst, the Geometry Center at the University of Minnesota, and Berlin. Since 2006 he has been working at the GeometrieWerkstatt at the University of Tübingen.

Nicholas Schmitt

Nicholas Schmitt研究常平均曲率曲面的微分几何以及它们的构造和分类。他的很多研究都涉及用于实验研究此类曲面的软件的开发。Schmitt在1993年从马萨诸塞大学获得博士学位。在阿默斯特中心、明尼苏达大学几何中心和柏林继续他对几何、 分析、 数值和图像的研究 。自 2006 年以来，他一直在杜宾根大学 的 Geometrie Werkstatt 工作。

DIFFERENTIAL GEOMETRY

Differential geometry is the study of curved shapes, like curved lines or surfaces in space, or even shapes with many dimensions. One studies questions like: What is the difference between the ways an egg and a cooling tower are curved? How can curvature be measured?

Of special interest are surfaces that are “optimal” in a precise sense, which means that a certain measure of “quality” cannot be improved by making small changes to the shape.

For example, soap films obtained by dipping a curved wire into soap solution have the least possible area among all surfaces spanning the wire. Soap bubbles minimize surface area among all closed surfaces enclosing a fixed volume of air. From the viewpoint of differential geometry the shapes of soap films and soap bubbles are singled out by the fact that the so-called “mean curvature” is the same at every point of the surface.

Several pictures in this gallery show surfaces of constant mean curvature, one picture shows a surface that does not have minimal area but minimal “total curvature”. Another picture shows a three dimensional space of constant curvature.

微分几何

微分几何研究弯曲的图形，比如空间中的曲线和曲面或者是高维的图形。 研究类似的问题： 鸡蛋和冷却塔的弯曲的方式之间的区别是什么？如何衡量这种弯曲？

特别感兴趣的曲面是在某种意义下是最好的。意思是说某种"质量"的度量不能通过小更改形状来得到改进。

例如，把弯曲的金属丝放进肥皂水里得到的肥皂薄膜在所有张成这个金属丝的曲面中拥有最小的面积。肥皂泡在所有含有固定量空气的封闭曲面中表面积最小。从微分几何的观点看，肥皂薄膜和肥皂泡都属于一类所谓的"平均曲率"在曲面上每一点处都是相同的曲面。

此展览中的几个图片展示的都是常平均曲率曲面，其中一张图片展示的曲面不具有极小的面积但具有极小的“全曲率”，另外一张图片展示的是一个三维的常曲率空间。

Twizzle Torus

The Twizzle Torus is an annular surface with a constant mean curvature in the three-dimensional sphere, a space curved in itself. To make it visible it must first be projected into our flat space. Luckily, this is enabled such that basic shape features can be maintained. In the three-dimensional sphere where it belongs to it has a screw symmetry which can still be surmised during projection.

The Twizzle Torus is only a comparatively simple example in an endless hierarchy of increasingly complex annular surfaces with similar curvature features.

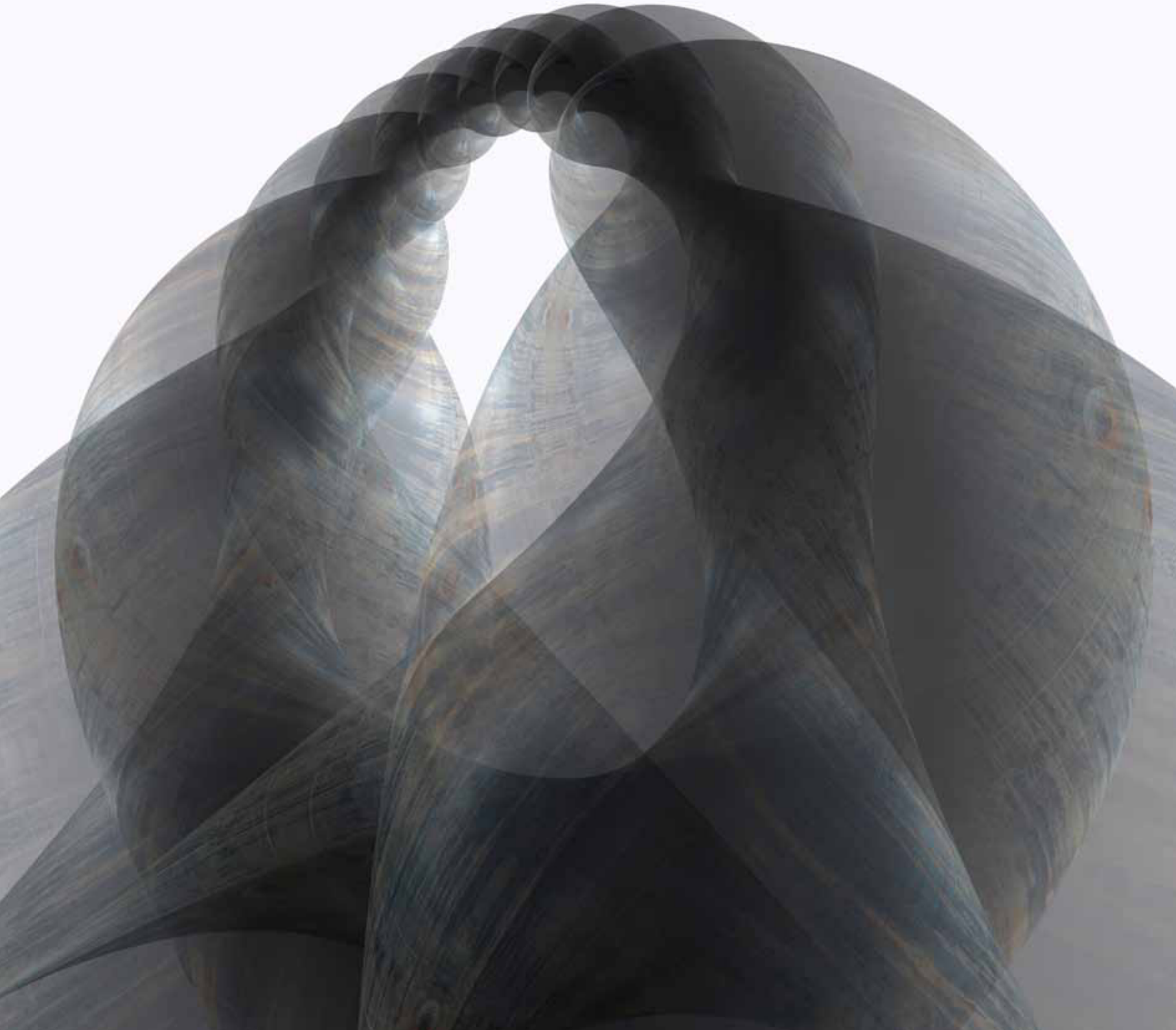
The surface was developed by Nicholas Schmitt (GeometrieWerkstatt Tübingen) and he has designed and calculated the image using XLab software.

Twizzle 圆环

Twizzle 圆环面是三维球面（一个弯曲的空间）中具有常曲率的环形曲面。使其可见它必须首先被投影到我们的平坦空间中。幸运的是，维持基其本的形状特征是可行的。在三维球面上，它具有螺旋对称性，在投影过程中仍然可以被推测出来。

Twizzle 环面只是在具有类似曲率特征的无穷层次结构的复环面中相对较简单的。

这个曲面由Nicholas Schmitt (GeometrieWerkstatt 图宾根大学)开发，他使用 XLab 软件设计和计算这个图像。



DISCRETE MINIMAL SURFACE

Minimal surfaces are a classical subject matter of differential geometry. They are surfaces whose mean curvature vanishes everywhere. The best known among this surface class are the Catenoid and the Helicoid. One of the many interesting features of minimal surfaces is the existence of an associated family. This means that the surfaces are deformed in such a way that they remain minimal (they are even isometrical to each other and corresponding tangent planes are parallel). Helicoid and Catenoid belong to the same associated family. As a result, they can be deformed into one another and in between surfaces are minimal, as well.

The image shows a discretization of the minimal surface half way between Catenoid and Helicoid. It is composed of spheres and circles touching each other at their points of contact. There is also an associated family for these discrete minimal surfaces and both the radii of corresponding spheres and the positions of corresponding circle disks are equal.

The scene was developed with ¡Reality, the image itself was calculated by POV- Ray.

离散的极小曲面

极小曲面是微分几何中的经典课题。它们是平均曲率处处为零的曲面。悬链面和螺旋面是这类曲面中最著名的。极小曲面的有趣特征之一是每个极小曲面都存在一族相伴曲面。意思是它们在形变过程中仍然保持极小性（它们甚至是相互等距的，并且相应的切平面是平行的）。悬链面和螺旋面是属于同一族的。所以，它们可以通过形变互相转化，并且在它们形变过程中的曲面也是极小的。

图中展示的是悬链面和螺旋面中间的极小曲面的离散化。它是由相互接触的球面和圆组成的。这些离散的极小曲面也具有一族相伴曲面，相应球面的半径和相应圆盘的位置是相等的。

这个场景是由iReady设计出来的。图形的本身是由POV-Ray计算出来的。